Davies-trees in infinite combinatorics

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Introduction

Topic: infinite combinatorics.

- simple results (no background in set theory needed),
- common feature: the proofs are smart inductions:
  - list your objectives/obstacles,
  - life is hard = infinitely many obstacles,
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The first applications

- (**R. O. Davies, 1962**) $\mathbb{R}^2$ is **covered** by countably many **rotated graphs of functions**.
- (**S. Jackson, R. D. Mauldin, 2002**) There is a **subset A of $\mathbb{R}^2$** which intersects each isometric copy of $\mathbb{Z} \times \mathbb{Z}$ in **exactly one point**.
  - I.e. **each rotation A' of A tiles the plane**:

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A little bit about infinities

- smallest infinite sets: $\mathbb{N}$, $\mathbb{Q}$, ... **countable sets**,
- larger infinite sets: $\mathcal{P}(\mathbb{N})$, $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$ ... **uncountable sets**,
  - size of $\mathbb{R}$ is called the **continuum**,
  - **Continuum Hypothesis** (CH): the successor of $|\mathbb{N}|$ is the continuum,
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There is an **infinite** family $\mathcal{A}$ of **pairwise disjoint** infinite sets in $\mathbb{N}$:

$$A_i = \{p_i^k : k \in \mathbb{N}\} \text{ where } p_i \text{ is the ith prime.}$$

- Are there **uncountably many**? No.
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Hint: we can look at $\mathbb{Q}$ **instead of** $\mathbb{N}$:

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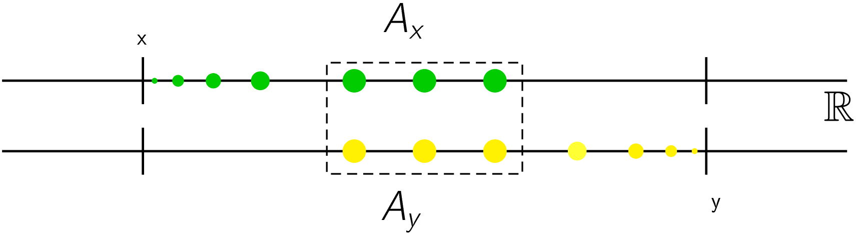

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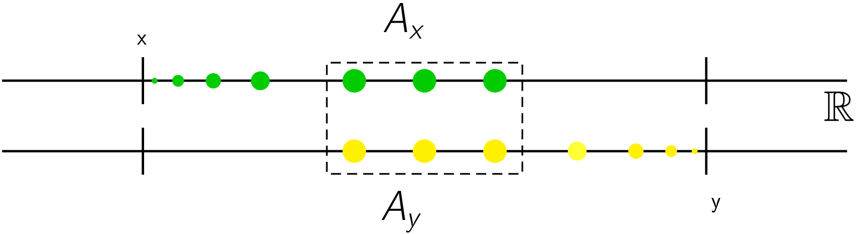

- $\{A_x : x \in \mathbb{R}\}$ is almost disjoint and has size continuum.

Degrees of disjointness

Definition

*A family $\mathcal{A}$ is **n-almost disjoint** iff $A \cap B$ **has size** $< n$ for all $A, B \in \mathcal{A}$.*

- Are there uncountable n-almost disjoint families in $\mathbb{N}$?

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*A family $\mathcal{A}$ is **essentially disjoint** iff there are finite $F_A \subseteq A$ such that $\{A \setminus F_A : A \in \mathcal{A}\}$ **is pairwise disjoint**.*

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Theorem (P. Komjáth, 1984)

Every n-almost disjoint family $\mathcal{A}$ is essentially disjoint (for any $n \in \mathbb{N}$).

- if $\mathcal{A}$ is countable then easy: $F_{A_j} = A_j \cap \bigcup_{i<j} A_i$ for $j \in \mathbb{N}$,
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- We have **no real recursive constructions** for almost disjoint families.

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Is there an almost disjoint family $\mathcal{A}$ on $\mathbb{N}$ such that for every $f : \mathbb{Q} \to \mathbb{N}$ there is $A \in \mathcal{A}$ with $f^{-1}(A)$ somewhere dense?

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In particular, **cyles of length 4** embed into G.

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Given that G has **large chromatic number**, are there **highly connected** subgraphs with large chromatic number?

- We **don't know** if one can always find **infinitely connected** subgraphs.
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We will cover the **plane** $\mathbb{R}^2$ with seemingly small sets.

Definition

A set $A \subseteq \mathbb{R}^2$ is a ***cloud around a point*** *$a \in \mathbb{R}^2$ iff A intersects every line L through a in a finite set.*

How many clouds can cover the plane?

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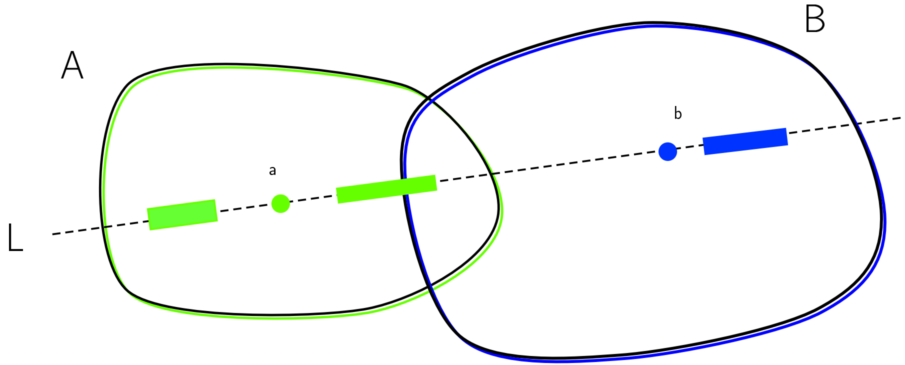

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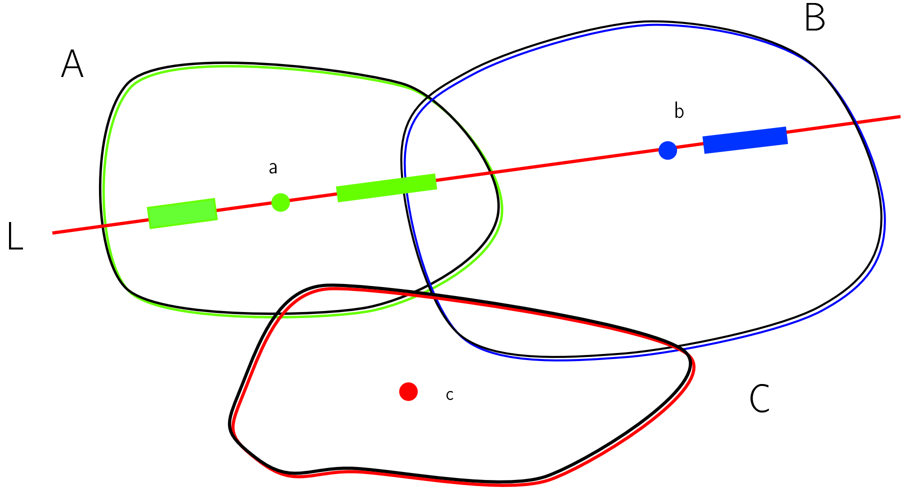

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The following are ***equivalent****:*

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Thank you very much for your attention!

"*The infinite we do now, the finite will have to wait a little.*"

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