Trees, ladders and graphs

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Topics - an informal overview

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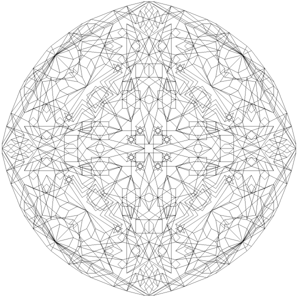

The general problem

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*

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The first results

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Uncountable chromatic number

What graphs must occur as subgraph of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are $\triangle$-free graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

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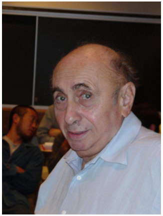

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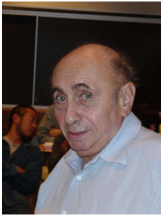

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Chromatic number and connectivity

[Erdős, Hajnal, 1966] K_{n,ω_1} embeds into G if $Chr(G) > \omega$.

K_{n,ω_1} is **n-connected**, hence the question: suppose that $Chr(G) > \omega$, is it true that

- there is an n-**connected uncountably chromatic** subgraph?
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- **[Komjáth, 1988]** Under PFA, if $|G| = Chr(G) = \omega_1$ then G **has an infinitely connected** uncountably chromatic subgraph.

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Theorem (D.S. 2014)

*There is a graph of **chromatic number** ω_1 and size continuum **without uncountable infinitely connected subgraphs**.*

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Another obligatory graph from **[Hajnal-Komjáth, 1984]**:

- If $Chr(G) > \omega$ then $H_{\omega,\omega+1}$ **embeds** into G.
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Let's do some set theory!

"*The infinite we do now, the finite will have to wait a little.*"

P. Erdős