# University of Toronto
## Faculty of Arts and Science

MAT136H1F Calculus I (B)
Term Test
Fall 2014
Duration: 90 minutes
No Aids Allowed

Family Name: UTIONS

Given Name: SOL

Student Number: $\infty$

Lecture and Tutorial:

| LEC 0101 | LEC 5101 | TUT 0101 | TUT 0201 | TUT 5101 | TUT 5201 |
|---|---|---|---|---|---|
| MWF9 | R6-9 | M3 | R4 | T5 | R5 |

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

- **Organize your work.** Write your answers in the space provided. Work scattered over the page without clear ordering will receive very little credit.
- **Justify your answers.** An incorrect answer supported by correct calculations and explanations might still receive partial credit.
- **Need more space?** If you need more space, use the backs of the pages and clearly indicate when you have done this. You can also use the backs of pages for rough-work.

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

1. Evaluate the definite and indefinite integrals below.

(a) (4 points) $\int (2t+1)^7 dt$.

$= \int u^7 \left(\frac{du}{2}\right)$ $\quad u = 2t+1 \quad du = 2\,dt$

$= \frac{1}{2}\int u^7\,du$

$= \frac{1}{16}u^8 + C$

$= \frac{1}{16}(2t+1)^8 + C$

(b) (3 points) $\int_0^{\pi/2} \sin(\cos(x))\sin(x)dx$.

$u = \cos x$
$du = -\sin x\,dx$

$= \int_{u(0)}^{u(\pi/2)} \sin u \ \sin x \left(\frac{du}{-\sin x}\right)$

$= -\int_1^0 \sin u\,du$

$= +\int_0^1 \sin u\,du$

$= [-\cos u]_0^1 = -\cos(1) + \cos(0) = 1 - \cos(1)$.

(c) (3 points) $\int x\sqrt{x-2}\,dx$.

$u = \sqrt{x-2}$
$u^2 = x-2 \quad u^2+2 = x$
$2u\,du = dx$

$= \int (u^2+2)u\,(2u\,du)$

$= 2\int (u^4 + 2u^2)\,du$

$= 2\int u^4 du + 4\int u^2 du$

$= \frac{2}{5}u^5 + \frac{4}{3}u^3 + C = \frac{2}{5}(\sqrt{x-2})^5 + \frac{4}{3}(\sqrt{x-2})^3 + C$

2. Evaluate the definite and indefinite integrals below.

(a) (3 points) $\int \frac{1}{x^2-1}dx$.

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\frac{1}{x^2-1} = \frac{A(x+1)+B(x-1)}{x^2-1} \qquad \frac{1}{x^2-1} = \frac{(A+B)x+(A-B)}{x^2-1}$$

$A+B=0$
$A-B=1$
$A=\frac{1}{2}$
$B=-\frac{1}{2}$

$$\int \frac{1}{x^2-1}dx = \int\left(\frac{\left(\frac{1}{2}\right)}{x-1} + \frac{\left(-\frac{1}{2}\right)}{x+1}\right)dx = \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1}$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$$

$$= \ln\left|\sqrt{\frac{x-1}{x+1}}\right| + C$$

(b) (3 points) $\int_0^{\pi/2} \cos^2(x)\sin^3(x)dx$.

First find $\int \cos^2(x)\sin^3(x)dx = \int \cos^2(x)\left(1-\cos^2(x)\right)\sin x\,dx$

$u=\cos(x)$
$du=-\sin(x)dx$

$$= \int u^2(1-u^2)\left(\sin x\left(\frac{du}{-\sin x}\right)\right)$$

$$= -\int u^2(1-u^2)\,du = \int(u^4-u^2)\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C.$$

$$\int_0^{\pi/2}\cos^2(x)\sin^3(x)dx = \left[\frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x)\right]_0^{\pi/2} = \frac{1}{5}\left[\cos^5(x)\right]_0^{\pi/2} - \frac{1}{3}\left[\cos^3(x)\right]_0^{\pi/2}$$

$$= \frac{1}{5}(0-1) - \frac{1}{3}(0-1) = \frac{-1}{5} + \frac{1}{3} = \frac{2}{15}.$$

(c) (4 points) $\int \sqrt{12 - 4s - s^2}ds$.

$$12 - 4s - s^2 = -(s^2 + 4s - 12) = -\left( s^2 + 4s + 4 - 4 - 12 \right) = -\left( (s+2)^2 - 16 \right)$$
$$= 16 - (s+2)^2$$

$$\int \sqrt{12 - 4s - s^2}\, ds = \int \sqrt{16 - (s+2)^2} = 4 \int \sqrt{1 - \left(\frac{s+2}{4}\right)^2}\, ds$$

$u = \frac{s+2}{4}$

$du = \frac{1}{4} ds$

$$= 16 \int \sqrt{1 - u^2}\, du \qquad u = \cos\theta \quad du = -\sin\theta\, d\theta$$

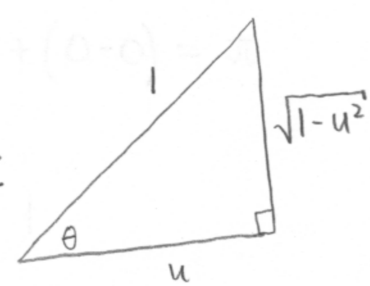
$$= -16 \int \sqrt{1 - \cos^2\theta}\, \sin\theta\, d\theta = -16 \int \sin^2\theta\, d\theta$$

$$= -16 \int \left( \frac{1}{2} - \frac{1}{2}\cos(2\theta) \right) d\theta = -8 \int d\theta + 8 \int \cos(2\theta)\, d\theta$$

$$= -8\theta + 4\sin(2\theta) + C$$

$$= -8 \arccos(u) + 8u\sqrt{1-u^2} + C$$

$$= -8 \arccos\left(\frac{s+2}{4}\right) + 2(s+2)\sqrt{1 - \left(\frac{s+2}{4}\right)^2} + C$$




$\cos\theta = u$

$\sin\theta = \sqrt{1-u^2}$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2u\sqrt{1-u^2}$

3. (a) (2 points) Determine whether $f(x) = x\sin(x)$ is an even function, odd function, both, or neither.

$$f(-x) = (-x)(\sin(-x)) = (-x)(-\sin(x)) = x\sin(x) = f(x)$$

So $f(x)$ is an even function.

(b) (2 points) Write the average value of $f(x)$ on the interval $[0, \pi]$ as an integral.

$$f_{av} = \frac{1}{\pi - 0}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx$$

(c) (4 points) Find the average value of $f(x)$ on the interval $[0, \pi]$.

$$\int_0^{\pi} \underbrace{x}_{u}\,\underbrace{\sin x\,dx}_{dv} = \Big[\underbrace{x}_{u}\underbrace{(-\cos x)}_{v}\Big]_0^{\pi} - \int_0^{\pi}\underbrace{(-\cos x)}_{v}\,\underbrace{dx}_{du} = -\big[x\cos x\big]_0^{\pi} + \int_0^{\pi}\cos x\,dx$$

$$= -\big[x\cos x\big]_0^{\pi} + \big[\sin x\big]_0^{\pi} = -(-\pi - 0) + (0 - 0) = \pi$$

$$f_{av} = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{1}{\pi}(\pi) = 1$$

(d) (2 points) Find the average value of $g(x) = \sin(x^{17})$ on the interval $[-2\pi, 2\pi]$.

$$g(-x) = \sin((-x)^{17}) = \sin(-x^{17}) = -\sin(x^{17}) = -g(x)$$

$g(x)$ is an odd function

$$g_{av} = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} g(x)\,dx = \frac{1}{4\pi}(0) = 0$$

4. Suppose a car travels on a straight road away from Toronto. Let $p(t)$ be its position relative to Toronto (measured in $km$) at time $t$ (measured in $hr$). Also suppose its velocity (in $km/hr$) at time $t$ is denoted by $v(t)$. Let's assume we have information about the car's velocity, i.e. we know that the function $v(t)$ is given by $v(t) = t^2e^t$.

   (a) (5 points) Determine the net change in the position of the car from $t = 0$ to $t = 3$.

$$p(3) - p(0) \overset{\text{FTC}}{=} \int_0^3 v(t)\,dt = \int_0^3 t^2e^t\,dt = \left[t^2e^t - 2te^t + 2e^t\right]_0^3$$

$$\int \underbrace{t^2}_{u}\underbrace{e^t dt}_{dv} = \underbrace{t^2}_{u}\underbrace{e^t}_{v} - \int \underbrace{e^t}_{v}\underbrace{(2t\,dt)}_{du} = t^2e^t - 2\int \underbrace{t}_{u}\underbrace{e^t dt}_{dv} = t^2e^t - 2\left(\underbrace{t}_{u}\underbrace{e^t}_{v} - \int \underbrace{e^t}_{v}\underbrace{dt}_{du}\right)$$

$$= t^2e^t - 2te^t + 2e^t + C$$

$$= (3)^2e^3 - 2(3)e^3 + 2e^3 - \left(0^2e^0 - 2(0)e^0 + 2e^0\right) = 9e^3 - 6e^3 + 2e^3 - 2$$

$$= 5e^3 - 2 \text{ kms} \quad (\approx 5(3)^3 - 2 = 133 \text{ kms})$$

   (b) (5 points) Define $F(x) = \int_x^{x^2} \frac{\sin(s)}{s}\,ds$ for $x \geq 1$. Find $\frac{d}{dx}F(x)$.

$$F(x) = \int_x^2 \frac{\sin(s)}{s}\,ds + \int_2^{x^2} \frac{\sin(s)}{s}\,ds = -\int_2^x \frac{\sin(s)}{s}\,ds + \int_2^{x^2} \frac{\sin(s)}{s}\,ds$$

Define $H(y) = \int_2^y \frac{\sin(s)}{s}\,ds$ then $H'(y) \overset{\text{FTC}}{=} \frac{\sin(y)}{y}$.

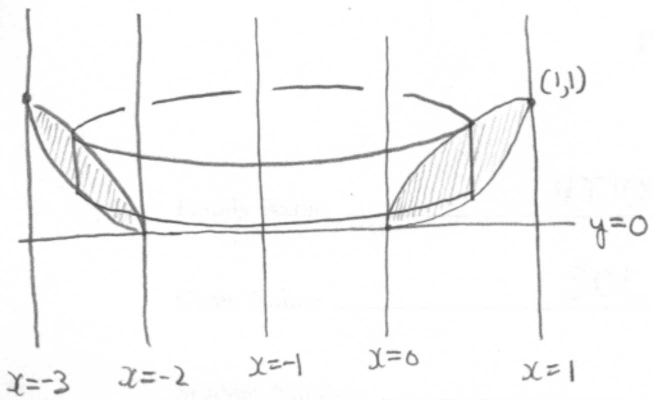
$$F(x) = -H(x) + H(x^2)$$

$$F'(x) = -H'(x) + H'(x^2)\cdot(2x)$$

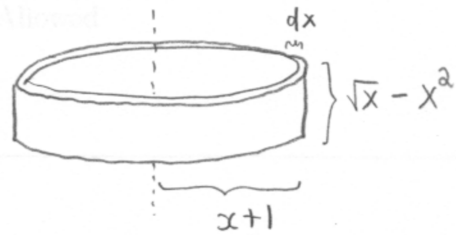
$$= -\frac{\sin x}{x} + \frac{\sin(x^2)}{x^2}(2x) = \frac{2\sin(x^2) - \sin(x)}{x}.$$

5. Consider the region $R$ between the two curves $y = \sqrt{x}$ and $y = x^2$ from $x = 0$ to $x = 1$.

(a) (5 points) Consider the shape created by rotating the region $R$ around the line $x = -1$. Let its volume be denoted by $V_1$. Write $V_1$ as an integral, and explain the method of approximation that gave you the integral.



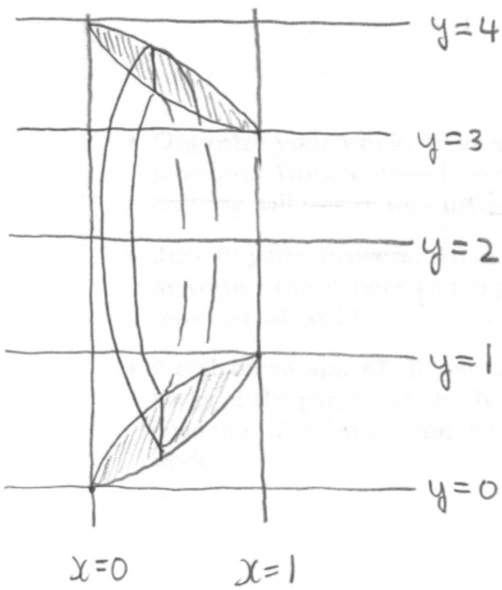

Cylindrical shells method



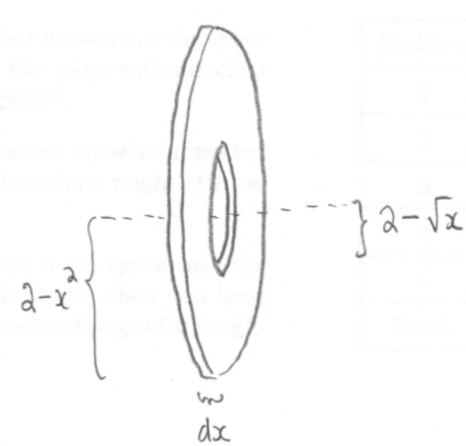

volume $= 2\pi(x+1)(\sqrt{x} - x^2)\,dx$

$$V_1 = \int_0^1 2\pi(x+1)(\sqrt{x} - x^2)\,dx$$

(b) (5 points) Consider the shape created by rotating the region $R$ around the line $y = 2$. Let its volume be denoted by $V_2$. Write $V_2$ as an integral, and explain the method of approximation that gave you the integral.




Washer method




volume $= [\pi(\text{radius}_{out})^2 - \pi(\text{radius}_{in})^2]$

$= \pi((2-x^2)^2 - (2-\sqrt{x})^2)d$

$$V_2 = \int_0^1 \pi\left[(2-x^2)^2 - (2-\sqrt{x})^2\right]dx$$