University of Toronto

Faculty of Arts and Science
Term Test (June 1 2015)
MAT135H1F - Calculus I (A)
Duration - 2 hours
No Aids Allowed

Family Name: ____________________

First Name: ____________________

Student Number: ____________________

Lecture and Tutorial section:

L0101	**L5101**	**T0101**	**T0201**	**T5101**	**T5102**
TR10-1	**TR6-9**	**TR1**	**TR2**	**TR5**	**TR5**

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work!** Write your answers in the space provided. Work scattered over the page without clear ordering will receive very little credit.
- **Justify your answers!** A correct answer without explanation or algebraic work will receive no credit; an incorrect answer supported by correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	85	

1. Answer the following short questions on limits.

   (a) (3 points) Find $\lim_{t\to\pi} \ln(\sin(t) - \cos(t))$.

   Solution: Note that the function $f(x) = \ln(x)$ is continuous at $x = 1$. Furthermore $\lim_{t\to\pi} \sin(t) = 0$ and $\lim_{t\to\pi} \cos(t) = -1$. Hence $\lim_{t\to\pi} \ln(\sin(t) - \cos(t)) = \ln(1) = 0$.

   (b) (3 points) Give an example of a function $f(x)$ such that $\lim_{x\to 1^-} f(x) > \lim_{x\to 1^+} f(x)$.

   Solution: You can take the function $f(x) = \dfrac{1-x}{|1-x|}$. Another way to write this function is

   $$f(x) = \begin{cases} 1 & \text{when } x < 1, \\ -1 & \text{when } x > 1. \end{cases}$$

   (c) (4 points) Find the following limit $\lim_{x\to 0} \dfrac{e^{2x}-1}{x}$ (Hint: express the limit as a derivative).

   Solution: Let $f(x) = e^{2x}$ Then $\lim_{x\to 0} \dfrac{e^{2x}-1}{x} = \lim_{x\to 0} \dfrac{f(x)-f(0)}{x} = f'(0) = 2$.
   (Note that $f'(x) = 2e^{2x}$ by Chain Rule)

(d) (5 points) Find $\lim_{x\to\infty} \sqrt{x^4+1} - (x^2+1)$.

Solution: $\lim_{x\to\infty} \sqrt{x^4+1} - (x^2+1) = \lim_{x\to\infty} (\sqrt{x^4+1} - (x^2+1)) \dfrac{\sqrt{x^4+1}+(x^2+1)}{\sqrt{x^4+1}+(x^2+1)}$

$= \lim_{x\to\infty} \dfrac{x^4+1-(x^2+1)^2}{\sqrt{x^4+1}+(x^2+1)}$

$= \lim_{x\to\infty} \dfrac{-2}{\sqrt{1+x^{-2}}+1+x^{-2}} = -1$

(e) (5 points) Find $\lim_{\alpha\to 0} \dfrac{\sin(\alpha)}{3\alpha+\tan(\alpha)}$.

Solution: We now that

$$\lim_{\alpha\to 0} \frac{\sin(\alpha)}{\alpha}$$

and also

$$\lim_{\alpha\to 0} \frac{\tan(\alpha)}{\alpha} = 1.$$

So

$$\lim_{\alpha\to 0} \frac{\sin(\alpha)}{3\alpha+\tan(\alpha)} = \lim_{\alpha\to 0} \frac{\frac{\sin(\alpha)}{\alpha}}{3+\frac{\tan(\alpha)}{\alpha}} = \frac{1}{3+1} = \frac{1}{4}.$$

2. Answer the following short questions on derivatives.

(a) (3 points) Find $\frac{d}{dx}\tan(x^2+1)$.

Solution: Using Chain Rule we get

$$\frac{d}{dx}\tan(x^2+1) = \sec^2(x^2+1)2x.$$

(b) (3 points) Find $\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right)$.

Solution: We apply the quotient rule:

$$\frac{d}{dx}\left(\frac{\sin(x)}{x^2}\right) = \frac{\cos(x)x^2 - 2x\sin(x)}{x^4}.$$

(c) (3 points) Are there any horizontal tangent lines to $f(x) = x\ln(x)$?

Solution: We need to find all points where $f'(x) = 0$. Note that $f'(x) = 1 \cdot \ln(x) + x\frac{1}{x} = \ln(x) + 1$. This implies that $\ln(x) = -1$ so $x = e^{-1} = \frac{1}{e}$.

(d) (3 points) Find $\frac{d}{dx}\left(\cos^2(x)\sin(x)\right)$.

Solution: $\frac{d}{dx}\left(\cos^2(x)\sin(x)\right) = 2\cos(x)(-\sin(x))\sin(x) + \cos^2(x)\cos(x)$
$= -2\cos(x)\sin^2(x) + \cos^3(x)$.

(e) (3 points) What is the 13th derivative of $x^{12} - x^2 + 1$?

Solution: Note that $f'(x) = 12x^{11} - 2x$ and $f^{(2)}(x) = (12)(11)x^{12} - 2$ and one can see that $f^{(12)}(x) = (12)(11)...(3)(2)(1)$ is constant. Therefore $f^{(13)}(x) = 0$.

3. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \frac{4-x^2}{x^3-8} & \text{when } x > 2, \\ \frac{c-x}{x+1} & \text{when } x \leq 2. \end{cases}$$

(a) (5 points) Find the values of c which makes $f(x)$ continuous.

Solution: $f(x)$ is continuous exactly if $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$.
Now

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{(2-x)(2+x)}{(x-2)(x^2+2x+4)} = -\frac{1}{3}$$

hence $f(2) = \frac{c-2}{2+1} = -\frac{1}{3}$ which implies that $c = 1$.

(b) (5 points) Working with the same function as in part (a), find the horizontal asymptotes of $f(x)$.

Solution: Note that

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{4-x^2}{x^3-8} = \lim_{x\to +\infty} \frac{\frac{4}{x^3} - \frac{1}{x}}{1 - \frac{8}{x^3}} = 0$$

and

$$\lim_{x\to -\infty} f(x) = \lim_{x\to +\infty} \frac{c-x}{x+1} = \lim_{x\to +\infty} \frac{\frac{c}{x} - 1}{1 + \frac{1}{x}} = -1.$$

Thus $y = 0$ and $y = -1$ are the two horizontal asymptotes of $f(x)$.

4. (a) (5 points) State the Intermediate Value Theorem.

Solution: IVT: Suppose that $f(x)$ is continuous on the interval $[a, b]$ and N is a number between $f(a)$ and $f(b)$. Then there is a c in (a, b) such that $f(c) = N$.

(b) (5 points) Use the Intermediate Value Theorem to show that the equation

$$x2^x = 2 + x$$

has at least two solutions. (Hint: one solution is positive and the other is negative.)

Solution: Let $f(x) = x2^x - (2 + x)$ and note that $f(x)$ is a continuous function. Furthermore $f(1) = -1 < 0$ and $f(2) = 2 > 0$ thus by IVT theorem $f(x) = 0$ for some $x \in (1, 2)$.

Also, $f(-1) = \frac{3}{2} > 0$ and $f(-4) < 0$ so again by the IVT there exists a root in the interval $(-4, -1)$.

5. (10 points) Find the derivative of $f(x) = \sqrt{1+x^3}$ at $x = 1$ using the definition of the derivative as a limit.

Solution: By the definition of the derivative

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{1+(1+h)^3} - \sqrt{2}}{h}.$$

To calculate the limit we use the usual trick

$$\lim_{h\to 0} \frac{\sqrt{1+(1+h)^3} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{\sqrt{1+(1+h)^3} - \sqrt{2}}{h} \frac{\sqrt{1+(1+h)^3} + \sqrt{2}}{\sqrt{1+(1+h)^3} + \sqrt{2}}$$

$$= \lim_{h\to 0} \frac{(1+h)^3 - 1}{h\sqrt{1+(1+h)^3} + \sqrt{2}} = \lim_{h\to 0} \frac{3 + 3h + h^2}{\sqrt{1+(1+h)^3} + \sqrt{2}} = \frac{3\sqrt{2}}{4}.$$

6. Let f and g be differentiable functions whose values at $x = 0, 1, 2$ are given in the table below:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	4	0	0	3
1	0	1	3	5
2	2	4	1	1

(a) (2 points) What is the equation of the tangent line to the graph of $f(x)$ at $x = 2$?

Solution:
The equation of the tangent line is given by $y = mx + d$ where $m = f'(2) = 4$. Hence $y = 4x + d$ and furthermore we know that the line should pass through the point $(2, 2)$ i.e. $2 = 2 \cdot 4 + d$. Thus $d = -6$ and the equation of the tangent is $y = 4x - 6$

(b) (2 points) Suppose $h(x) = x^2 f(x)$ and find $h'(2)$.

Solution: $h'(x) = 2xf(x) + x^2 f'(x)$ Hence $h'(2) = 4f(2) + 4f'(2) = 24$

(c) (3 points) Find $\lim\limits_{x \to 0} f\big(\frac{\sin(x)}{x}\big) g(2 + x)$.

Solution: Note that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ and f and g are continuous at 1 and 2 respectively so we obtain

$$\lim_{x \to 0} f\Big(\frac{\sin(x)}{x}\Big) g(2 + x) = f(1)g(2) = 0.$$

(d) (3 points) Suppose $h(x) = g(f(2x))$ and find $h'(1)$.

Solution: $h'(x) = g'(f(2x))f'(2x)2$ by applying the chain rule twice.
Hence $h'(1) = g'(f(2))f'(2)2 = 8$.

7. (10 points) Find the limit $\lim_{x\to 0} 2^{\sin\left(\frac{1}{x}\right)}\sqrt{x+x^2}$.

Solution: Note that $-1 < \sin(\frac{1}{x}) < 1$ and since the exponential function is strictly increasing we deduce that

$$\frac{1}{2} < 2^{\sin(\frac{1}{x})} < 2$$

and so multiplying both sides by $\sqrt{x+x^2}$ gives

$$2^{-1}\sqrt{x+x^2} < 2^{\sin\left(\frac{1}{x}\right)}\sqrt{x+x^2} < 2\sqrt{x+x^2}.$$

Now we have $\lim_{x\to 0}\sqrt{x+x^2} = 0$ and hence by Squeeze Theorem we deduce that

$$\lim_{x\to 0} 2^{\sin\left(\frac{1}{x}\right)}\sqrt{x+x^2} = 0.$$