THE INCONVENIENT D-PROPERTY

A solution to a problem

Dániel Soukup

Second year, MSc in Pure Mathematics

Email: daniel.t.soukup@gmail.com

Supervisor: Zoltán Szentmiklóssy, assistant professor

Eötvös Loránd University

November 2010, Budapest

Contents

1 Introduction **2**

2 Definitions **3**

3 Notes on MAD families **4**

4 Fragments of Shelah's club guessing **4**

5 The general construction **5**

6 Focusing on property D and aD **7**

7 Examples of aD, non linearly D-spaces **10**

8 Acknowledgements **11**

Appendix A **11**

Appendix A.1 Motivation for D-spaces 11

Appendix A.2 Relationship to covering properties 12

Appendix B **13**

1. Introduction

The notion of a *D-space* was probably first introduced by van Douwen and since than, many work had been done in this topic. Investigating the properties of D-spaces and the connections between other covering properties led to the definition of *aD-spaces*, defined by Arhangel'skii in [2]. As it turned out, property aD is much more docile then property D. In [3] Arhangel'skii asked the following, as one of the "most intriguing problems in the theory of D and aD-spaces":

Problem 4.6. Is there a Tychonoff aD-space which is not a D-space?

A negative answer to this question would settle almost all of the questions about the relationship of classical covering properties to property D. Quite similarly, Guo and Junnila in [8] asked the following about a weakening of property D:

Problem 2.12. Is every aD-space linearly D?

In G. Gruenhage's survey on D-spaces [7], another version of this question is stated (besides the original Arhangel'skii), namely:

Question 3.6(2) Is every scattered, aD-space a D-space?

The main result of this paper is the following answer to the questions above.

Theorem 1.1. *There exists a 0-dimensional T_2 space X such that X is scattered, aD and non linearly D.*

In [15] the author showed that the existence of a locally countable, locally compact space X of size ω_1 which is aD and non linearly D is independent of ZFC. Here we refine those methods and using Shelah's club guessing theory we answer the above questions in ZFC.

The paper has the following structure. In Sections 2, 3 and 4 we gather all the necessary facts about D-spaces, MAD families and club guessing. In Section 5 we define spaces $X[\lambda, \mu, \mathcal{M}, \underline{C}]$, where λ and $\mu = cf(\mu)$ are cardinals, $\mathcal{M}$ is a MAD family on μ and $\underline{C}$ is a guessing sequence. It is shown in Claim 5.2 that

(0) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is always T_2, 0-dimensional and scattered.

Section 6 contains two important results:

(1) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is not linearly D if $cf(\lambda) \geq \mu$ (see Corollary 6.3),

(2) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is aD under certain assumptions (see Corollary 6.9).

Finally in Section 7 we show how to produce such spaces $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ depending on the cardinal arithmetic and using Shelah's club guessing.

Although the paper is self-contained, we attach two appendices. In Appendix A, we present a few more facts about D and aD-spaces and explain why the problem under consideration is relevant. In Appendix B, we give a bit more detailed explanation of guessing sequences.

The reader is supposed to be familiar with the basic notions and notations of set-theory and general topology. However, for undefined terms and notations see [9] and [6], respectively.

2. Definitions

An *open neighborhood assignment* (ONA, in short) on a space (X, τ) is a map $U : X \to \tau$ such that $x \in U(x)$ for every $x \in X$. A space X is said to be a *D-space* if for every neighborhood assignment U, one can find a closed discrete $D \subseteq X$ such that $X = \bigcup_{d \in D} U(d) = \bigcup U[D]$ (such a set D is called a *kernel for* U). In [2] the authors introduced property aD:

Definition 2.1. *A space (X, τ) is said to be aD iff for each closed $F \subseteq X$ and for each open cover $\mathcal{U}$ of X there is a closed discrete $A \subseteq F$ and $\phi : A \to \mathcal{U}$ with $a \in \phi(a)$ for all $a \in A$ such that $F \subseteq \cup\phi[A]$.*

It is clear that D-spaces are aD. Proving that a space is aD, the notion of an *irreducible space* will play a key role. A space X is *irreducible* iff every open cover $\mathcal{U}$ has a *minimal open refinement* $\mathcal{U}_0$; meaning that no proper subfamily of $\mathcal{U}_0$ covers X. In [3] Arhangel'skii showed the following equivalence.

Theorem 2.2 ([3, Theorem 1.8])**.** *A T_1-space X is an aD-space if and only if every closed subspace of X is irreducible.*

Another generalization of property D is due to Guo and Junnila [8]. For a space X a cover $\mathcal{U}$ is *monotone* iff it is linearly ordered by inclusion.

Definition 2.3. *A space (X, τ) is said to be* linearly D *iff for any ONA $U : X \to \tau$ for which $\{U(x) : x \in X\}$ is monotone, one can find a closed discrete set $D \subseteq X$ such that $X = \bigcup U[D]$.*

We will use the following characterization of linear D property. A set $D \subseteq X$ is said to be $\mathcal{U}$*-big* for a cover $\mathcal{U}$ iff there is no $U \in \mathcal{U}$ such that $D \subseteq U$.

Theorem 2.4 ([8, Theorem 2.2])**.** *The following are equivalent for a T_1-space X:*

1. *X is linearly D.*
2. *For every non-trivial monotone open cover $\mathcal{U}$ of X, there exists a closed discrete $\mathcal{U}$-big set in X.*

We encourage the reader to look up Appendix A for a more detailed (and less dry) introduction to D-spaces. We also recommend G. Gruenhage's recently finished survey on D-spaces [7], summarizing the facts and the work done in the topic, stating numerous open problems.

3. Notes on MAD families

As MAD families will play an essential part in our constructions we observe some easy facts about them. Let μ be any infinite cardinal. We call $\mathcal{M} \subseteq [\mu]^\mu$ an *almost disjoint family* if $|M \cap N| < \mu$ for all distinct $M, N \in \mathcal{M}$. $\mathcal{M}$ is a *maximal almost disjoint family* (in short, a *MAD family*) if for all $A \in [\mu]^\mu$ there is some $M \in \mathcal{M}$ such that $|A \cap M| = \mu$.

We will use the following rather trivial combinatorial fact.

Claim 3.1. *Let $\mathcal{M} \subseteq [\mu]^\mu$ be a MAD family and $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$. Suppose that $N \in [\mu]^\mu$ and $|N \setminus \cup\mathcal{M}'| = \mu$ for all $\mathcal{M}' \in [\mathcal{M}]^{<\mu}$. Then $|\Phi| > \mu$ for $\Phi = \{\varphi < \kappa : |N \cap M^\varphi| = \mu\}$.*

Proof. If $|\Phi| < \mu$ then with $\widetilde{N} = N \setminus \bigcup\{M^\varphi : \varphi \in \Phi\} \in [\mu]^\mu$ we can extend the MAD family, which is a contradiction. If $|\Phi| = \mu$ then let $\Phi = \{\varphi_\zeta : \zeta < \mu\}$. By transfinite induction, construct $\widetilde{N} = \{n_\xi : \xi < \mu\}$ such that $n_\xi \in N \setminus (\bigcup\{M^{\varphi_\zeta} : \zeta < \xi\} \cup \{n_\zeta : \zeta < \xi\})$ for $\xi < \mu$. It is straightforward that $\widetilde{N} \notin \mathcal{M}$ and $\mathcal{M} \cup \{\widetilde{N}\}$ is almost disjoint, which is a contradiction. $\square$

From our point of view the sizes of MAD families are important. Clearly there is a MAD family on ω of size 2^ω. The analogue of this does not always hold for ω_1. Baumgartner in [4] proves that it is consistent with ZFC that there is no almost disjoint family on ω_1 of size 2^{ω_1}. However, we have the following fact.

Claim 3.2. *If $2^\omega = \omega_1$ then there is a MAD family $\mathcal{M}$ on ω_1 of size 2^{ω_1}.*

In Section 7 we use *nonstationary MAD families* $\mathcal{M}_{NS} \subseteq [\mu]^\mu$ meaning that $\mathcal{M}_{NS}$ is a MAD family such that every $M \in \mathcal{M}_{NS}$ is nonstationary in μ. Observe, that using Zorn's lemma to almost disjoint families of nonstationary sets of μ we can get nonstationary MAD families.

4. Fragments of Shelah's club guessing

The constructions of the upcoming sections will use the following amazing results of Shelah. For a cardinal λ and a regular cardinal μ let S^λ_μ denote the ordinals in λ with cofinality μ. For an $S \subseteq S^\lambda_\mu$ an *S-club sequence* is a sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that $C_\delta \subseteq \delta$ is a club in δ of order type μ.

Theorem 4.1 ([13, Claim 2.3]). *Let λ be a cardinal such that $cf(\lambda) \geq \mu^{++}$ for some regular μ and let $S \subseteq S^\lambda_\mu$ stationary. Then there is an S-club sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that for every club $E \subseteq \lambda$ there is $\delta \in S$ (equivalently, stationary many) such that $C_\delta \subseteq E$.*

A detailed proof of Theorem 4.1 can be found in [1, Theorem 2.17].

Theorem 4.2 ([14, Claim 3.5])**.** *Let λ be a cardinal such that $\lambda = \mu^+$ for some uncountable, regular μ and $S \subseteq S^\lambda_\mu$ stationary. Then there is an S-club sequence $\underline{C} = \langle C_\delta : \delta \in S \rangle$ such that $C_\delta = \{\alpha^\delta_\zeta : \zeta < \mu\} \subseteq \delta$ and for every club $E \subseteq \lambda$ there is $\delta \in S$ (equivalently, stationary many) such that:*

$$\{\zeta < \mu : \alpha^\delta_{\zeta+1} \in E\} \text{ is stationary.}$$

For a detailed proof, see [16]. We recommend Appendix B for the reader who first encounters guessing sequences as a brief explanation to this phenomenon.

5. The general construction

Definition 5.1. *Let $\lambda > \mu = cf(\mu)$ be infinite cardinals. Let $\mathcal{M} \subseteq [\mu]^\mu$ be a MAD family, $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$ and let $\underline{C} = \{C_\alpha : \alpha \in S^\lambda_\mu\}$ denote an S^λ_μ-club sequence. We define a topological space $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$ as follows. The underlying set of our topology will be a subset of the product $\lambda \times \kappa$. Let*

- $X_\alpha = \{(\alpha, 0)\}$ *for* $\alpha \in \lambda \setminus S^\lambda_\mu$,
- $X_\alpha = \{\alpha\} \times \kappa$ *for* $\alpha \in S^\lambda_\mu$,
- $X = \bigcup\{X_\alpha : \alpha < \lambda\}$.

Let $C_\alpha = \{a^\xi_\alpha : \xi < \mu\}$ denote the increasing enumeration for $\alpha \in S^\lambda_\mu$. For each $\alpha \in S^\lambda_\mu$ let

- $I^\xi_\alpha = (a^\xi_\alpha, a^{\xi+1}_\alpha]$ *for* $\xi \in succ(\mu) \cup \{0\}$,
- $I^\xi_\alpha = [a^\xi_\alpha, a^{\xi+1}_\alpha]$ *for* $\xi \in \lim(\mu)$.

Note that $\bigcup\{I^\xi_\alpha : \xi < \mu\} = (a^0_\alpha, \alpha)$ is a disjoint union.

Define the topology on X by neighborhood bases as follows;

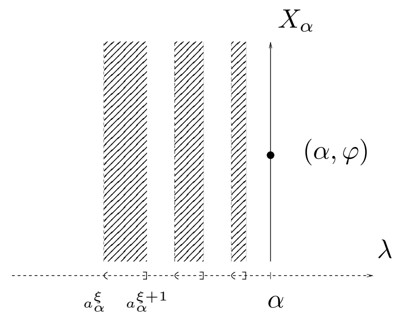
(i) for $\alpha \in S^\lambda_\mu$ and $\varphi < \kappa$ let

$$U((\alpha, \varphi), \eta) = \{(\alpha, \varphi)\} \cup \bigcup\{X_\gamma : \gamma \in \cup\{I^\xi_\alpha : \xi \in M^\varphi \setminus \eta\}\} \text{ for } \eta < \mu$$

and let

$$B(\alpha, \varphi) = \{U((\alpha, \varphi), \eta) : \eta < \mu\}$$

be a base for the point (α, φ).



(ii) for $\alpha \in S^\lambda_{<\mu} \cup succ(\lambda) \cup \{0\}$ *let* $(\alpha, 0)$ *be an isolated point,*

(iii) for $\alpha \in S^\lambda_{\mu'}$ *where* $\mu' > \mu$ *let*

$$U(\alpha,\beta) = \bigcup\{X_\gamma : \beta < \gamma \leq \alpha\} \text{ for } \beta < \alpha$$

and let

$$B(\alpha) = \{U(\alpha,\beta) : \beta < \alpha\}$$

be a base for the point $(\alpha, 0)$.

It is straightforward to check that these *basic open sets* form neighborhood bases.

⋆

Fix some cardinals $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ and S^λ_μ-club sequence $\underline{C}$. In the following $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$.

Claim 5.2. *The space* $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ *is 0-dimensional,* T_2 *and scattered. Observe that*

(a) X_α *is closed discrete for all* $\alpha < \lambda$*, moreover*

(b) $\bigcup\{X_\alpha : \alpha \in A\}$ *is closed discrete for all* $A \in [\lambda]^{<\mu}$,

(c) $X_{\leq\alpha} = \bigcup\{X_\beta : \beta \leq \alpha\}$ *is clopen for all* $\alpha < \lambda$.

Proof. First we prove that $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is T_2. Note that

$(*)$ $\bigcup\{X_\gamma : \beta < \gamma \leq \alpha\}$ is clopen for all $\beta < \alpha < \lambda$.

Thus $(\alpha, \varphi), (\alpha', \varphi') \in X$ can be separated trivially if $\alpha \neq \alpha'$. Suppose that $\alpha = \alpha' \in S^\lambda_\mu$ and $\varphi \neq \varphi' < \kappa$. There is $\eta < \mu$ such that $(M^\varphi \cap M^{\varphi'}) \setminus \eta = \emptyset$ since $|M^\varphi \cap M^{\varphi'}| < \mu$. Thus $U((\alpha,\varphi),\eta) \cap U((\alpha,\varphi'),\eta) = \emptyset$.

Next we show that $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is 0-dimensional. By $(*)$ it is enough to prove that $U((\alpha,\varphi),\eta)$ is closed for all $\alpha \in S^\lambda_\mu$, $\varphi < \kappa$ and $\eta < \mu$. Suppose $x = (\alpha', \varphi') \in X \setminus U((\alpha,\varphi),\eta)$, we want to separate x from $U((\alpha,\varphi),\eta)$ by an open set. Let $\alpha = \alpha'$. There is $\eta' < \mu$ such that $(M^{\varphi} \cap M^{\varphi'}) \setminus \eta' = \emptyset$, thus $U((\alpha,\varphi),\eta) \cap U((\alpha,\varphi'),\eta') = \emptyset$. Let $\alpha \neq \alpha'$. If $\alpha' \in S^{\lambda}_{<\mu} \cup \mathrm{succ}(\lambda) \cup \{0\}$ then x is isolated, thus we are done. Suppose $\alpha \in S^{\lambda}_{\mu'}$ where $\mu' \geq \mu$. Then $\beta = \sup(C_\alpha \setminus \alpha') < \alpha'$ thus $U(\alpha',\beta) \cap U((\alpha,\varphi),\eta) = \emptyset$.

$X[\lambda,\mu,\mathcal{M},\underline{C}]$ is scattered since $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is right separated by the lexicographical ordering on $\lambda \times \kappa$.

(a) and (c) is trivial, we prove (b). Suppose $x = (\alpha',\varphi') \in X$, we prove that there is a neighborhood U of x such that $|U \cap \bigcup\{X_\alpha : \alpha \in A\}| \leq 1$. If $\alpha' \in S^{\lambda}_{<\mu} \cup \mathrm{succ}(\lambda) \cup \{0\}$ then x is isolated, thus we are done. Suppose $\alpha \in S^{\lambda}_{\mu'}$ where $\mu' \geq \mu$. Then $\beta = \sup(A \setminus \alpha') < \alpha'$ thus the open set $U = \{x\} \cup \bigcup\{X_\gamma : \beta < \gamma < \alpha\}$ will do the job. □

6. Focusing on property D and aD

Again fix some cardinals $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^{\varphi} : \varphi < \kappa\} \subseteq [\mu]^{\mu}$ and S^{λ}_{μ}-club sequence $\underline{C}$. Our next aim is to investigate the spaces $X = X[\lambda,\mu,\mathcal{M},\underline{C}]$ concerning property D and aD.

Definition 6.1. *Let $\pi(F) = \{\alpha < \lambda : F \cap X_\alpha \neq \emptyset\}$ for $F \subseteq X$. F is said to be* (un)bounded *if $\pi(F)$ is (un)bounded in λ.*

Claim 6.2. *If $F \subseteq X$ and $\pi(F)$ accumulates to $\alpha \in S^{\lambda}_{\eta}$ such that $\mu \leq \eta < \lambda$ then $F' \cap X_\alpha \neq \emptyset$.*

Proof. If $\eta > \mu$ then $X_\alpha = \{(\alpha,0)\}$ and each neighborhood $U(\alpha,\beta)$ of $(\alpha,0)$ intersects F. Thus $F' \cap X_\alpha \neq \emptyset$. Let us suppose that $\pi(F)$ accumulates to $\alpha \in S^{\lambda}_{\mu}$. Since $\bigcup\{I^{\xi}_{\alpha} : \xi < \mu\} = (a^0_\alpha, \alpha)$, the set $N = \{\xi < \mu : I^{\xi}_{\alpha} \cap \pi(F) \neq \emptyset\}$ has cardinality μ. Thus there is some $\varphi < \kappa$ such that $|N \cap M^{\varphi}| = \mu$, since $\mathcal{M}$ is MAD family. It is straightforward that $(\alpha,\varphi) \in F'$ since $U((\alpha,\varphi),\eta) \cap F \neq \emptyset$ for all $\eta < \mu$. □

Corollary 6.3. *If $cf(\lambda) \geq \mu$ then a closed unbounded subspace $F \subseteq X$ is not a linearly D-subspace of X. Hence $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is not a linearly D-space.*

Proof. Let $F \subseteq X$ be closed unbounded. $|\pi(D)| < \mu$ for every closed discrete $D \subseteq X$ by Claim 6.2. Thus there is no big closed discrete set for the open cover $\{X_{\leq\alpha} : \alpha < \lambda\}$ which shows that F is not linearly D by Theorem 2.4. □

Our aim now is to prove that in certain cases the space $X[\lambda,\mu,\mathcal{M},\underline{C}]$ is an aD-space, equivalently every closed subspace of it is irreducible; see Theorem 2.2.

Claim 6.4. *Every closed, bounded subspace $F \subseteq X$ is a D-subspace of X; hence F is irreducible.*

Proof. We prove that $F \subseteq X$ is a D-subspace of X by induction on $\alpha = \sup \pi(F) < \lambda$. Let $U : F \to \tau$ be an ONA. If α is a successor (or $\alpha = 0$), then $F_0 = F \setminus U((\alpha,0))$ is closed and $\sup(F_0) < \alpha$ thus we are easily done by induction.

Let $\alpha \in S^\lambda_{\mu'}$ where $\mu \leq \mu' < \lambda$. Then $\sup \pi(F_0) < \alpha$ where $F_0 = F \setminus \cup U[X_\alpha \cap F]$ by Claim 6.2. Thus we are easily done by induction and the fact that X_α is closed discrete.

Now let $\nu = cf(\alpha) < \mu$, let $\sup\{\alpha_\xi : \xi < \nu\} = \alpha$ such that $\alpha_0 = 0$ and $\{\alpha_\xi : \xi < \nu\}$ is strictly increasing. Let $J_\xi = \bigcup\{X_\gamma : \alpha_\xi \leq \gamma \leq \alpha_{\xi+1}\}$ if $\xi < \nu$ is limit or $\xi = 0$ and $J_\xi = \bigcup\{X_\gamma : \alpha_\xi < \gamma \leq \alpha_{\xi+1}\}$ if $\xi < \nu$ is a successor. Let $J_\nu = X_\alpha$. Clearly $\{J_\xi : \xi \leq \nu\}$ is a discrete family of disjoint clopen sets such that $\bigcup\{J_\xi : \xi \leq \nu\} = X_{\leq\alpha}$. $F = \bigcup\{F^\xi : \xi \leq \nu\}$ where $F^\xi = F \cap J_\xi$ is closed for $\xi \leq \nu$. By induction, for all $\xi < \nu$ there is some closed discrete kernel $D^\xi \subseteq F^\xi$ for the restriction of U to F^ξ. Let $D^\nu = F^\nu$. Then $D = \bigcup\{D^\xi : \xi \leq \nu\}$ is closed discrete and $F \subseteq \cup U[D]$. □

To handle the unbounded closed subsets we need the following definition.

Definition 6.5. *Let $F_\alpha = F \cap X_\alpha$ for $F \subseteq X$ and $\alpha < \lambda$. A* subset $F \subseteq X$ is high enough *if*

$$|\{\alpha < \lambda : |F_\alpha| = |F|\}| \geq \mu.$$

We say that a subset $F \subseteq X$ is high *if every closed unbounded subset of F is high enough.*

The following rather technical claim will be useful.

Claim 6.6. *For any $F \subseteq X$ and ONA $U : F \to \tau$ such that $U(x)$ is a basic open neighborhood of $x \in F$, let*

$$Y_F = \{x \in F : \exists\alpha < \lambda : F_\alpha \subseteq U(x), |F_\alpha| = |F|\},$$

$$\Gamma_F = \{\alpha < \lambda : |F_\alpha| = |F|, \exists x \in F : F_\alpha \subseteq U(x)\}.$$

If F is closed and high enough then $Y_F, \Gamma_F \neq \emptyset$.

Proof. Since $Y_F \neq \emptyset$ iff $\Gamma_F \neq \emptyset$, it is enough to show that there is some $x \in Y_F$. Since F is high enough, $|Z| \geq \mu$ for $Z = \{\alpha' < \lambda : |F| = |F_{\alpha'}|\}$. Let $D = \bigcup\{F_{\alpha'} : \alpha' \in Z\} \subseteq F$. Let $\beta \in S^\lambda_\mu$ be an accumulation point of $Z = \pi(D)$. Then by Claim 6.2 there is some $x \in D' \cap X_\beta$ thus $x \in F$. Clearly $x \in Y_F$. □

Theorem 6.7. *If the closed unbounded $F \subseteq X$ is high then F is irreducible.*

Proof. Suppose that $\mathcal{U}$ is an open cover of F. We can suppose that we refined it to the form $\{U(x) : x \in F\}$ where each $U(x)$ is basic open. From Claim 6.6 we know that $Y_F, \Gamma_F \neq \emptyset$. We define $Y^\xi \subseteq F$ by induction.

- Let $\alpha_0 \in \Gamma_F$ and $Y^0 = \{x \in Y_F : F_{\alpha_0} \subseteq U(x)\}$. Fix some $h^0 : Y^0 \to F_{\alpha_0}$ injection; this exists because $|F_{\alpha_0}| = |F| \geq |Y_F| \geq |Y^0|$.
- Suppose we defined $\alpha_\zeta < \lambda$ and Y^ζ for $\zeta < \xi$. Let

  $$F^\xi = F \setminus \left(\bigcup\{U(x) : x \in \cup\{Y^\zeta : \zeta < \xi\}\} \cup X_{\leq\alpha}\right)$$

  where $\alpha = \sup\{\alpha_\zeta : \zeta < \xi\}$.

- If F^ξ is bounded then stop. Notice that F_ξ is bounded iff $F \setminus \bigcup\{U(x) : x \in \cup\{Y^\zeta : \zeta < \xi\}\}$ is bounded.
- Suppose F^ξ is unbounded. $F^\xi \subseteq F$ is closed either thus F^ξ is high enough since F is high. Hence $Y_{F^\xi}, \Gamma_{F^\xi} \neq \emptyset$.
- Let $\alpha_\xi \in \Gamma_{F^\xi}$; thus $|F^\xi_{\alpha_\xi}| = |F^\xi|$ and $F^\xi_{\alpha_\xi}$ is covered by some $U(x)$ for $x \in F^\xi$. Let $Y^\xi = \{x \in Y_{F^\xi} : F^\xi_{\alpha_\xi} \subseteq U(x)\}$. Fix some $h^\xi : Y^\xi \to F^\xi_{\alpha_\xi}$ injection; this exists because $|F^\xi_{\alpha_\xi}| = |F^\xi| \geq |Y_{F^\xi}| \geq |Y^\xi|$.

Lemma 6.8. *The induction stops before μ many steps.*

Proof. Suppose we defined this way $\{\alpha_\xi : \xi < \mu\}$ and let $\alpha = \sup\{\alpha_\xi : \xi < \mu\} \in S^\lambda_\mu$. Let $D = \bigcup\{F_{\alpha_\xi} : \xi < \mu\}$. By Claim 6.2 there is some $x \in D' \cap X_\alpha$, thus $x \in F$ either. Clearly $F_{\alpha_\xi} \subseteq U(x)$ for μ many $\xi < \mu$. By the definition of the induction

$(*)$ for every $\zeta < \xi < \mu$ and every $y \in Y^\zeta$: $F^\xi_{\alpha_\xi} \cap U(y) = \emptyset$

Clearly by $(*)$, $x \notin Y^\zeta$ for all $\zeta < \mu$ since there is $\zeta < \xi < \mu$ such that $F^\xi_{\alpha_\xi} \subseteq U(x)$. Moreover $x \notin U(y)$ for every $y \in Y^\zeta$ and $\zeta < \mu$; if $x \in U(y)$ then since $x \neq y$ there is some $\beta < \alpha$ such that $\bigcup\{X_\gamma : \beta < \gamma \leq \alpha\} \subseteq U(y)$. This contradicts $(*)$ since there is $\zeta < \xi < \mu$ such that $\beta < \alpha_\xi$, thus $F^\xi_{\alpha_\xi} \subseteq U(y)$. Thus $x \in F^\xi$ for all $\xi < \mu$. Then $x \in Y^\xi$ for all $\xi < \mu$ such that $F_{\alpha_\xi} \subseteq U(x)$. This is a contradiction. ∎

Thus let us suppose that the induction stopped at step $\xi < \mu$, meaning that $\widetilde{F} = F \setminus \bigcup\{U(x) : x \in Y\}$ is bounded where $Y = \cup\{Y^\zeta : \zeta < \xi\}$. Let $h = \bigcup\{h^\zeta : \zeta < \xi\}$, $h : Y \to F$ is a 1-1 function since the sets $\mathrm{dom}(h^\zeta) = Y^\zeta$ and $ran(h^\zeta) \subseteq F^\zeta_{\alpha_\zeta}$ are pairwise disjoint for $\zeta < \xi$. Note that $\mathrm{ran}(h) \subseteq \bigcup\{F_{\alpha_\zeta} : \zeta < \xi\}$ is closed discrete by Claim 5.2. For $x \in Y$ let

$$U_0(x) = (U(x) \setminus \mathrm{ran}(h)) \cup \{h(x)\},$$

note that $U_0(x)$ is open. Then

$$\bigcup\{U_0(x) : x \in Y\} = \bigcup\{U(x) : x \in Y\}$$

is a minimal open refinement, since $h(x)$ is only covered by $U_0(x)$ for all $x \in Y$. Let $\mathcal{U}_0 = \{U_0(x) : x \in Y\}$

Let $V(x) = U(x) \setminus \bigcup\{F_{\alpha_\zeta} : \zeta < \xi\}$. Then $\mathcal{V} = \{V(x) : x \in \widetilde{F}\}$ is an open cover of $\widetilde{F}$, refining $\mathcal{U}$; $F_{\alpha_\zeta} \cap \widetilde{F} = \emptyset$ by construction for all $\zeta < \xi$. $\widetilde{F}$ is closed and bounded thus irreducible by Claim 6.4, hence there is an irreducible open refinement $\mathcal{V}_0$ of $\mathcal{V}$. It is straightforward that $\mathcal{V}_0 \cup \mathcal{U}_0$ is a minimal open refinement of $\mathcal{U}$ covering F. ∎

Corollary 6.9. *Suppose that $\lambda > \mu = cf(\mu)$ are infinite cardinals such that $cf(\lambda) \geq \mu$. Let $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ be a MAD family and $\underline{C}$ an S^λ_μ-club sequence. If $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high then $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is a 0-dimensional, Hausdorff, scattered space which is aD however not linearly D.*

Proof. $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is 0-dimensional, Hausdorff and scattered by Claim 5.2 and not linearly D by Corollary 6.3. It suffices to show that every closed $F \subseteq X$ is irreducible. If F is bounded then F is a D-space by Claim 6.4 hence irreducible. If F is unbounded, then F is high since X is high. Hence F is irreducible by Theorem 6.7. □

7. Examples of aD, non linearly D-spaces

In this section we give examples of aD, non linearly D-spaces of the form $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$. First let us make an observation.

Claim 7.1. *If $C_\alpha \subseteq \pi(F)'$ for a closed $F \subseteq X$ and $\alpha \in S^\lambda_\mu$, then $F_\alpha = X_\alpha$.*

Proof. Clearly $\bigcup\{X_\gamma : \gamma \in I^\xi_\alpha\} \cap F \neq \emptyset$ for all $\xi < \mu$. Thus every point in X_α is an accumulation point of F, thus $F_\alpha = X_\alpha$ since F is closed. □

Corollaries 7.3 and 7.5 below give certain examples of high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ spaces.

Proposition 7.2. *Suppose that μ is a regular cardinal, $cf(\lambda) \geq \mu^{++}$. Let $\underline{C}$ be an S^λ_μ-club guessing sequence from Theorem 4.1. If $\mathcal{M} \subseteq [\mu]^\mu$ is a MAD family of size at least λ then $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high.*

Proof. Let $F \subseteq X$ closed, unbounded. Then $\pi(F)'$ is a club in λ, hence there exists a stationary $S \subseteq S^\lambda_\mu$ such that $C_\alpha \subseteq \pi(F)'$ for all $\alpha \in S$. Thus $F_\alpha = X_\alpha$ by Claim 7.1 hence $|F_\alpha| = |\mathcal{M}| = |X|$ for all $\alpha \in S$. □

Corollary 7.3.

1. *Suppose that $2^\omega \geq \omega_2$. Let $\mathcal{M}$ be a MAD family on ω of size 2^ω and let $\underline{C}$ be an $S^{\omega_2}_\omega$-club guessing sequence from Theorem 4.1. Then $X[\omega_2, \omega, \mathcal{M}, \underline{C}]$ is high.*
2. *Suppose that $2^\omega = \omega_1$ and $2^{\omega_1} \geq \omega_3$. Let $\mathcal{M}$ be a MAD family on ω_1 of size 2^{ω_1} (exists by Claim 3.2) and let $\underline{C}$ be an $S^{\omega_3}_{\omega_1}$-club guessing sequence from Theorem 4.1. Then $X[\omega_3, \omega_1, \mathcal{M}, \underline{C}]$ is high.*

Proposition 7.4. *Suppose that $\lambda = \mu^+ > \mu = cf(\mu) > \omega$ and let $\underline{C}$ be an $S^{\mu^+}_\mu$-club guessing sequence from Theorem 4.2. If there is a nonstationary MAD family $\mathcal{M}_{NS} \subseteq [\mu]^\mu$ such that $|\mathcal{M}_{NS}| = \mu^+$ then $X = X[\mu^+, \mu, \mathcal{M}_{NS}, \underline{C}]$ is high.*

Proof. Let $\mathcal{M}_{NS} = \{M^\varphi : \varphi < \mu^+\}$ and $\underline{C} = \langle C_\alpha : \alpha \in S^{\mu^+}_\mu \rangle$ such that $C_\alpha = \{a^\xi_\alpha : \xi < \mu\} \subseteq \alpha$. Suppose that the closed $F \subseteq X$ is unbounded. Then $\pi(F)'$ is a club in μ^+, hence there exists a stationary $S \subseteq S^{\mu^+}_\mu$ such that

$$N_\alpha = \{\xi < \mu : a^{\xi+1}_\alpha \in \pi(F)'\} \text{ is stationary in } \mu$$

for all $\alpha \in S$. Fix any $\alpha \in S$, we prove that $|F_\alpha| = |F|$. N_α is stationary so by applying Claim 3.1 we get that $|\Phi_\alpha| = \mu^+$ for $\Phi_\alpha = \{\varphi < \mu^+ : |N_\alpha \cap M^\varphi| = \mu\}$. Note that $F \cap \bigcup\{X_\gamma : \gamma \in I_\alpha^\xi\} \neq \emptyset$ for $\xi \in N_\alpha$. Thus (α, φ) is an accumulation point of F for $\varphi \in \Phi_\alpha$, hence $\{\alpha\} \times \Phi_\alpha \subseteq F_\alpha$. Thus $|F_\alpha| = \mu^+ = |X|$. ☐

Corollary 7.5. *Suppose that* $2^{\omega_1} = \omega_2$. *Let* $\underline{C}$ *be an* $S^{\omega_2}_{\omega_1}$*-club guessing sequence from Theorem 4.2 and let* $\mathcal{M}_{NS}$ *be a nonstationary MAD family on* ω_1. *Then* $X[\omega_2, \omega_1, \mathcal{M}_{NS}, \underline{C}]$ *is high.*

Thus, by all means we can deduce the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that in any model of ZFC, either $(2^\omega \geq \omega_2)$ or $(2^\omega = \omega_1 \wedge 2^{\omega_1} \geq \omega_3)$ or $(2^{\omega_1} = \omega_2)$. Using Corollaries 7.3 and 7.5 above, depending on the sizes of 2^ω and 2^{ω_1}, we see that there exists a high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ space. We are done by Corollary 6.9. ☐

8. Acknowledgements

The author would like to thank Assaf Rinot for his ideas and advices to look deeper into the theory of club guessing in ZFC.

Appendix A.

The aim of this section is to motivate the study of D-spaces and related properties, as well as to present a few facts about the relationship between standard covering properties and D, aD-spaces. We shall also see, that the above solved problem was worth studying.

Appendix A.1. Motivation for D-spaces

Compactness is one of the main concepts of general topology. A space X is *compact* iff every open cover of X has a finite subcover. Equivalently, if for every open neighborhood assignment $x \mapsto U(x)$, there is a finite subset $D \subseteq X$ such that $X = \cup U[D]$. Note, that if X is compact then $D \subseteq X$ is finite iff it is closed and discrete. Thus, the definition of D-spaces is a logical generalization of compactness.

Definition Appendix A.1. *A space* X *is said to be a* D-space *if for every neighborhood assignment* U*, one can find a closed discrete* $D \subseteq X$ *such that* $X = \bigcup_{d \in D} U(d) = \bigcup U[D]$

If we restrict our definition to monotone covers of a space X, we get the definition of linearly D-spaces, see Definition 2.3.

It is easy to see, that every compact space X is *irreducible*, that is, each open cover $\mathcal{U}$ of X has a minimal open refinement $\mathcal{U}_0$. Meaning, that there is no proper subfamily of $\mathcal{U}_0$ covering X. This observation is generalized with property aD. Indeed, from Theorem 2.2 we know that a T_1 space X is aD iff every closed subset $F \subseteq X$ is irreducible.

Now, it is straightforward to ask two things.

1. What is the connection between these covering properties?
2. How are they connected to classical covering properties?

Concerning the first question, it is easy to see that every D-space is aD and of course linearly D. The result of this paper is that the converse is not true. That is, by Theorem 1.1 there exists an aD-space which is not D or even linearly D. The second question is a harder one, and we discuss it in the next section.

Appendix A.2. Relationship to covering properties

The following are easy to see.

Proposition Appendix A.2.

1. *Every compact, moreover every σ-compact space is a D-space.*
2. *Every countably compact D-space is compact.*

Despite the work done in the topic by many great mathematicians, we lack theorems stating, that a classical covering property, that is fairly weaker than compactness, imply D. In fact, the following covering properties are not known to imply property D (even if you add "hereditarily"):
Lindelöf, paracompact, ultraparacompact, strongly paracompact, metacompact, metalindelöf, subparacompact, submetacompact, submetalindelöf, paralindelöf, screenable, σ-metacompact.

Actually, the problem, whether Lindelöf implies D, is the 14th of the twenty central problems in set-theoretic topology [12]. However, Arhangel'skii proved the following.

Theorem Appendix A.3 ([3, Theorem 1.15])**.** *Every submetalindelöf T_1 space is aD.*

Submetalindelöfness is a significant weakening of both Lindelöfness and paracompactness.

Definition Appendix A.4. *A space X is* submetalindelöf *iff for every open cover $\mathcal{U}$ of X there are $\{\mathcal{U}_n : n \in \omega\}$ open refinements of $\mathcal{U}$ covering X such that for every $x \in X$ there is $n \in \omega$ such that $|\{U \in \mathcal{U}_n : x \in U\}| \leq \omega$.*

Thus, until the recent results of this paper, there was the hope to prove $aD \Rightarrow D$ and settle the above hard-to-attack problems.

Let us cite another result.

Theorem Appendix A.5 ([8, Proposition 2.11])**.** *Every submetalindelöf T_1 space is linearly D.*

Thus, unfortunately, our constructions cannot be examples of non D-spaces satisfying some classical covering properties (since they are not linearly D). If there is such an example, it must be a non D-space with both property aD and linear D.

Despite all the open problems, there are fascinating results either. We take the opportunity again to recommend G. Gruenhage's survey on D-spaces [7], which contains all that we know and do not know about the topic.

Appendix B.

In the following, we give a very brief introduction to *guessing sequences*. Informally, a *guessing sequence* $\mathcal{S}$ *on a cardinal* κ is a family of subsets of κ, which in some way "guesses" all subsets of κ, while $|\mathcal{S}| < 2^\kappa$.

The first appearance of guessing sequences was in the following principle of Jensen [10], called *diamond*, denoted by $\diamondsuit$.

Definition Appendix B.1. $\diamondsuit$ *is the statement that there exists a sequence* $\mathcal{S} = \langle A_\alpha : \alpha < \omega_1 \rangle$ *of subsets of* ω_1 *such that for every* $A \subseteq \omega_1$ *there is some* $\alpha < \omega_1$ *such that* $A \cap \alpha = A_\alpha$.

Jensen discovered this principle while investigating Gödel's constructible universe and used $\diamondsuit$ to prove the existence of a combinatorial object, a Suslin-tree.

A well known weakening of Jensen's $\diamondsuit$ was introduced by Ostaszewski in [11], called the *club principle*, denoted by $\clubsuit$. It was used in [11] to define a topology on ω_1 with certain interesting properties.

Definition Appendix B.2. $\clubsuit$ *is the statement that there exists a sequence* $\mathcal{S} = \langle A_\alpha : \alpha < \omega_1 \rangle$ *such that*

1. $A_\alpha \subseteq \alpha$ *is an* ω*-type sequence cofinal in* α *for every* $\alpha \in \omega_1$,
2. *for every* $A \in [\omega_1]^{\omega_1}$ *there is some* $\alpha < \omega_1$ *such that* $A_\alpha \subseteq A$.

Since their introductions, these principles and their generalizations became very popular. They are commonly used to attack problems in set-theory or topology, as well as in measure theory or group theory.

However, both $\diamondsuit$ and $\clubsuit$ share the property that they are independent of the classical ZFC axioms of set theory. Meaning that there are models of set theory where $\diamondsuit$ holds (see Gödel's $V = L$), and there are models where $\diamondsuit$ fails (under Martin's axiom). The same is true for $\clubsuit$.

Therefore, it was a surprising result when Shelah came up with certain kinds of guessing sequences, which's existence can be proven from ZFC **only**. Section 4 contains the results important to us. The club guessing theory, when we only aim to guess club subsets, was used to prove coloring-theorems, in pcf theory or to prove the existence of Jónsson-algebras.

Although $\diamondsuit$-like principles are frequently used in general topology, there was a lack of direct applications of club-guessing till now; the only one found by the author is in [5].

References

[1] U. Abraham, M. Magidor, Cardinal Arithmetic – Handbook of Set Theory (Eds. Foreman, Kanamori), Volume 2, 1149-1229.

[2] A.V.Arhangel'skii and R. Buzyakova, Addition theorems and D-spaces, Comment. Mat. Univ. Car. 43(2002), 653-663.

[3] A.V. Arhangel'skii, D-spaces and covering properties, Topology and Appl. 146-147(2005), 437- 449.

[4] J. E. Baumgartner, Almost-disjoint sets, the dense set problem and the partition calculus, Ann. Math. Logic 9 (1976), no. 4, 401-439.

[5] T. Eisworth and P. Nyikos, First countable, countably compact spaces and the Continuum Hypothesis, Trans. Amer. Math. Soc. 357 (2005), 4329-4347.

[6] R. Engelking, General topology, 2nd ed., Heldermann Verl., Berlin, 1989.

[7] G. Gruenhage, A survey of D-spaces, to appear.

[8] H. Guo and H.J.K. Junnila, On spaces which are linearly D, Topology and Appl., Volume 157, Issue 1, 1 January 2010, Pages 102-107.

[9] T. Jech, Set theory, The Third Millennium Edition, Springer-Verlag 2002

[10] R. Bjorn Jensen, The fine structure of the constructible hierarchy, Ann. Math. Logic, 4:229-308; erratum, ibid. 4 (1972), 443, 1972. With a section by Jack Silver.

[11] A. J. Ostaszewski, On countably compact, perfectly normal spaces, J. London Math. Soc. (2), 14(3):505-516, 1976.

[12] Open Problems in Topology II, Edited by E. Pearl, Elsevier, 2007

[13] S. Shelah, Cardinal Arithmetic – Oxford Logic Guides, vol. 29, Oxford Univ. Press (1994).

[14] S. Shelah, Colouring and non-productivity of $\aleph_2$-c.c. – Annals Pure and Applied Logic 84 (1997) 153-174.

[15] D. Soukup, Properties D and aD Are Different, Top. Proc. 38 (2011) pp. 279-299.

[16] D. Soukup, L. Soukup, Club guessing for dummies, Arxiv note (arXiv: 1003.4670)