Modern techniques in combinatorial set theory

VO Advanced topics in mathematical logic

2019S.25073.1

March - June 2019

Course description

What original ideas lead to the breakthrough results of infinite combinatorics in the last 30 years? Which methods found the widest range of applications? The main purpose of this course is to overview novel techniques from the theory of (mostly) discrete structures of small uncountable size. We will survey applications from graph theory, the geometry of Euclidean spaces, topology, analysis and algebra. We will cover inductive constructions based on elementary submodels; coherent maps, walks on ordinals and applications of ρ-functions; and various approximation schemes. We shall also point out several open problems that wait to be solved. The course aims to provide a *working mathematician's toolbox* without going too deep into any one specific area.

Course and examination language: English.

Prerequisites

The course is aimed at advanced bachelor and graduate students with an interest in combinatorial questions, set theory or logic. There are no official prerequisites; however, we will assume familiarity with basic concepts of set theory e.g., ordinals and cardinals; stationary and club sets; transfinite induction; what is a formula, satisfaction and a model of a theory; basic concepts in graph theory and Ramsey's theorem on $\mathbb{N}$. See Chapter I and II from Kunen's textbook [5].

No background will be assumed regarding independence results, knowledge of forcing, model theory, or partition calculus.

Assessment

The final mark is composed of three components: participation (50%), submitted assignments (30%) and a 'presentation in pairs' component (20%).

- **Participation (50%):** you are expected to attend the lectures and encouraged to actively participate in class discussion.
- **Assignments (30%):** at each lecture, a number of **exercises** and **problems** will be announced (approx. 5 per lecture). You select the questions you like and submit solutions typed in Latex; the problems announced at a given lecture can be submitted for 2 weeks. Each correct solution for an exercise amounts to 0.5%, each problem to 1% of the maximal 30% that can be earned.[1]

[1] Submitting one exercise plus one problem per lecture will be sufficient to earn the maximal points. Incorrect submissions can be resubmitted to earn the mark. You are welcome to submit as many problems as you like, I will aim to provide feedback in due time.

- **Presentation in pairs (20%):** working in pairs, you will select a recent result/article closely related to the main topics of the course (plenty of recommendations will be provided). After understanding the material, you prepare a **joint 30-minute presentation** on the result. You should outline the context, main ideas and connections to the course material. Presentations will take place during the examination period.

Extra credit can be earned by attending (all or some of) the Advanced Class 2019[2] conference (June 26-29 2019, Vienna). Details will be announced later.

Lectures and office hours

Instructor: Daniel T. Soukup (daniel.soukup@univie.ac.at). Please use your university email address for class-related correspondence.

Lectures: Tuesday and Wednesday 13:00-14:00 at Room 101 in Josephinum (Wahringer Str. 25). Lectures start on March 5.

Office hours: Wednesday 14:00-15:30 at Room 91 or 101 in Josephinum (Wahringer Str. 25). You are encouraged to visit the office hours

- if you have any questions regarding the lectures,
- need hints or help with the exercises and problems,
- to discuss your final presentation topic and its preparations,
- if you feel like chatting about math or academics.

Website and discussion forum

Assignments, course information and all related course materials will be posted on Moodle. In addition to the office hours, you will have the chance to ask questions and discuss the topics on the Moodle forum.

Course topics in detail

We aim to cover the following three main topics.

1. the ubiquitous elementary submodels [9, 3, 4];
   (i) a detailed introduction to closure arguments and elementary submodels;
   (ii) simple applications: classical properties of set systems and infinite graphs;
   (iii) applications from the set theory of Euclidean spaces;
   (iv) Balogh's Q-set space construction: how to find non-trivial topological spaces in which any subset is a G_δ set?
2. coherent maps, minimal walks and some of their applications [10, 8];
   (i) introduction to ladder systems and coherent maps on ω_1;
   (ii) walks along ladder systems and simple properties of ρ-functions;
   (iii) on the structure of linear orders, constructing Countryman lines;
   (iv) an overview of Ramsey-type results on $\aleph_0, \aleph_1$ and $\aleph_2$.
3. general construction schemes [7, 2, 1, 6]
   (i) constructions along chains and trees of elementary submodels, paradoxical decompositions of the plane and the Steinhaus tiling problem;
   (ii) weak and strong guessing principles, old and new applications;
   (iii) Kurepa families and Todorcevic's construction scheme, more on coherent maps.

[2] See webpage here https://sites.google.com/view/estc2019/advanced-class-yst

References

[1] Uri Abraham and Stevo Todorčević. Partition properties of ω_1 compatible with CH. *Fund. Math.*, 152(2):165–181, 1997.

[2] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[3] András Hajnal and Péter Komjáth. What must and what need not be contained in a graph of uncountable chromatic number? *Combinatorica*, 4(1):47–52, 1984.

[4] Steve Jackson and R. Daniel Mauldin. Survey of the Steinhaus tiling problem. *Bull. Symbolic Logic*, 9(3):335–361, 2003.

[5] Kunen, Kenneth. Set theory an introduction to independence proofs. Vol. 102. Elsevier, 2014.

[6] Lopez, Fulgencio, and Stevo Todorcevic. ”Trees and gaps from a construction scheme.” Proceedings of the American Mathematical Society 145.2 (2017): 871-879.

[7] Justin Moore, Michael Hrušák, and Mirna Džamonja. Parametrized $\diamondsuit$ principles. *Transactions of the American Mathematical Society*, 356(6):2281–2306, 2004.

[8] Saharon Shelah. Colouring and non-productivity of $\aleph_2$-cc. *Annals of Pure and Applied Logic*, 84(2):153–174, 1997.

[9] Dániel T. Soukup and Lajos Soukup. Infinite combinatorics plain and simple. *to appear in the Journal of Symbolic Logic, arXiv preprint:1705.06195*, 2017.

[10] S. Todorcevic. Walks on ordinals and their characteristics. *Progress in Mathematics*, 263, 2007.