MAT136H1F – Quiz 3

TUT0101 – M3 (TA: I. Angelopoulos)

Fall, 2014

FAMILY NAME: ....

GIVEN NAME: ....

STUDENT ID: ....

Mark your lecture and tutorial sections:

L0101 (morning)	L5101 (evening)	T0101 (M3)	T0102 (R4)	T5101 (T5)	T5201 (R5)

You have 15 minutes to solve the problems. Each problem is worth 2 points. Good luck!

Question 1. *Consider the curve* $y = \ln(1-x^2)$ *for* $0 \le x \le 1/2$. *Write the arc length* L *as an integral.*

$$y'(x) = \frac{1}{1-x^2}(-2x) = \frac{-2x}{1-x^2} \qquad y'(x)^2 = \left(\frac{-2x}{1-x^2}\right)^2 = \frac{4x^2}{(1-x^2)^2}$$

$$L = \int_0^{1/2} \sqrt{1+y'(x)^2}\,dx = \int_0^{1/2} \sqrt{1+\frac{4x^2}{(1-x^2)^2}}\,dx$$

Question 2. *Express the improper integral* $\int_1^3 \frac{1}{2-x}dx$ *in terms of limits and definite integrals according to its definition.*

$$\int_1^3 \frac{dx}{2-x} \overset{def}{=} \lim_{b\to 2^-} \int_1^b \frac{dx}{2-x} + \lim_{b\to 2^+} \int_b^3 \frac{dx}{2-x}$$

Question 3. *Determine if* $\int_1^3 \frac{1}{2-x}dx$ *is convergent or divergent and evaluate if possible.*

Let's look at the part $\lim_{b\to 2^-} \int_1^b \frac{dx}{2-x} = \lim_{b\to 2^-} \Big[-\ln(2-x)\Big]_1^b$

$$= \lim_{b\to 2^-} \Big[-\ln(2-b) + \ln(2-1)\Big] = -\lim_{b\to 2^-} \ln(2-b) = \infty$$

Just from the divergence of one of the parts,

we can conclude that $\int_1^3 \frac{dx}{2-x}$ diverges.

MAT136H1F – Quiz 3

TUT0201 – R4 (TA: B. Navarro Lameda)

Fall, 2014

FAMILY NAME: ....

GIVEN NAME: ....

STUDENT ID: ....

Mark your lecture and tutorial sections:

L0101 (morning)	L5101 (evening)	T0101 (M3)	T0102 (R4)	T5101 (T5)	T5201 (R5)

You have 15 minutes to solve the problems. Each problem is worth 2 points. Good luck!

Question 1. *Consider the curve $y = 2x^{3/2}$ for $0 \leq x \leq 1$. Write the arc length L as an integral.*

$y'(x) = 3x^{1/2}$ $\quad y'(x)^2 = 9x$

$$L = \int_0^1 \sqrt{1 + y'(x)^2}\, dx = \int_0^1 \sqrt{1+9x}\, dx$$

Question 2. *Calculate the arc length L from Question 1.*

$$\int \sqrt{1+9x}\, dx = \int u \left(\frac{2}{9} u\, du\right) = \frac{2}{9}\int u^2 du$$

$$= \frac{2}{9}\left(\frac{1}{3}u^3\right) + C = \frac{2}{27}\left(\sqrt{1+9x}\right)^3 + C$$

$u = \sqrt{1+9x}$

$u^2 = 1+9x$

$\frac{u^2-1}{9} = x$

$\frac{2}{9} u\, du = dx$

$$L = \int_0^1 \sqrt{1+9x}\, dx = \left[\frac{2}{27}\left(\sqrt{1+9x}\right)^3\right]_0^1 = \frac{2}{27}\left(\sqrt{1+9}^{\,3} - \sqrt{1+9(0)}^{\,3}\right)$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right)$$

Question 3. *Express the improper integral $\int_0^1 \frac{1}{x\ln(x)} dx$ in terms of limits and definite integrals according to its definition. (You do not have to evaluate the integral.)*

$\frac{1}{x\ln(x)}$ has an asymptote at $x=0$ and $x=1$. Let $x=\frac{1}{2}$ be a point in between.

$$\int_0^1 \frac{dx}{x\ln(x)} \overset{\text{def}}{=} \lim_{b\to 0^+} \int_b^{1/2} \frac{dx}{x\ln(x)} + \lim_{b\to 1^-} \int_{1/2}^{b} \frac{dx}{x\ln(x)}$$

MAT136H1F – Quiz 3

TUT5101 – T5 (TA: A. Stewart)

Fall, 2014

FAMILY NAME: ....

GIVEN NAME: ....

STUDENT ID: ....

Mark your lecture and tutorial sections:

L0101 (morning)	L5101 (evening)	T0101 (M3)	T0102 (R4)	T5101 (T5)	T5201 (R5)

You have 15 minutes to solve the problems. Each problem is worth 2 points. Good luck!

Question 1. *Find a curve $y = f(x)$ such that the integral $\int_4^5 \sqrt{2 - 2x + x^2}dx$ is the arc length of the curve over some interval.*

need: $2 - 2x + x^2 = 1 + f'(x)^2 = 1 + (1 - 2x + x^2) = 1 + (1-x)^2$

equate: $f'(x) = 1 - x$ therefore $f(x) = x - \frac{x^2}{2}$ (take $C=0$).

the curve is $y = x - \frac{x^2}{2}$ the interval is $[4,5]$.

Question 2. *Express the improper integral $\int_0^\infty \frac{r}{e^r}dr$ in terms of limits and definite integrals according to its definition.*

$$\int_0^\infty \frac{r}{e^r}dr \overset{\text{def}}{=} \lim_{b\to\infty} \int_0^b \frac{r}{e^r}dr$$

Question 3. *Determine if $\int_0^\infty \frac{r}{e^r}dr$ is convergent or divergent and evaluate if possible.*

$$\int \frac{r}{e^r}dr = \int \underbrace{r}_{u}\underbrace{e^{-r}dr}_{dv} = \underbrace{-e^{-r}}_{v}\underbrace{r}_{u} - \int \underbrace{(-e^{-r})}_{v}\underbrace{dr}_{du} = -re^{-r} - e^{-r} + C$$

Therefore $\int_0^\infty \frac{r}{e^r}dr \overset{\text{def}}{=} \lim_{b\to\infty}\int_0^b \frac{r}{e^r}dr = \lim_{b\to\infty}\left[-re^{-r} - e^{-r}\right]_0^b = \lim_{b\to\infty}\left[-be^{-b} - e^{-b} - ((0)e^{-0} - e\right.$

$$= \lim_{b\to\infty}\left[\underbrace{\frac{-b}{e^b}}_{\downarrow\, 0} - \underbrace{\frac{1}{e^b}}_{\downarrow\, 0} + \underbrace{1}_{\downarrow\, 1}\right] = 1.$$

$\int_0^\infty \frac{r}{e^r}dr$ is convergent and equal to 1.

MAT136H1F – Quiz 3

TUT5201 – R5 (TA: B. Navarro Lameda)

Fall, 2014

FAMILY NAME: ....

GIVEN NAME: ....

STUDENT ID: ....

Mark your lecture and tutorial sections:

L0101 (morning)	L5101 (evening)	T0101 (M3)	T0102 (R4)	T5101 (T5)	T5201 (R5)

You have 15 minutes to solve the problems. Each problem is worth 2 points. Good luck!

Question 1. *Find a curve $y = f(x)$ such that the integral $\int_0^2 \sqrt{1+\frac{x^2}{x^4+2x^2+1}}\,dx$ is the arc length of the curve over some interval.*

need: $1+\dfrac{x^2}{x^4+2x^2+1} = 1+f'(x)^2 = 1+\left(\dfrac{x}{x^2+1}\right)^2$

I have omitted this calculation

equate: $f'(x)=\dfrac{x}{x^2+1}$ therefore $f(x)=\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$ (take $C=0$)

the curve is $y=\frac{1}{2}\ln(x^2+1)$ the interval is $[0,2]$

Question 2. *Express the improper integral $\int_0^\pi \tan(s)ds$ in terms of limits and definite integrals according to its definition.*

$$\int_0^\pi \tan(s)ds \overset{\text{def}}{=} \lim_{b\to\frac{\pi}{2}^-}\int_0^b \tan(s)ds + \lim_{b\to\frac{\pi}{2}^+}\int_b^\pi \tan(s)ds$$

Question 3. *Determine if $\int_0^\pi \tan(s)ds$ is convergent or divergent and evaluate if possible.*

look at first part: $\displaystyle\lim_{b\to\frac{\pi}{2}^-}\int_0^b \tan(s)ds = \lim_{b\to\frac{\pi}{2}^-}\Big[\ln(\sec(s))\Big]_0^b$

$$=\lim_{b\to\frac{\pi}{2}^-}\left[\ln\left(\frac{1}{\cos b}\right)-\ln\left(\frac{1}{\cos(0)}\right)\right] = \lim_{b\to\frac{\pi}{2}^-}\left[\ln\left(\frac{1}{\cos b}\right)\right] =$$

Just from the divergence of one of the parts, we can conclude that $\int_0^\pi \tan(s)ds$ diverges.