PROBLEM SET 6

DUE DATE: MAY 8, 2019

The ordinal ω_1 has a natural order topology. Basic open neighbourhoods of β are the half-open intervals $(\alpha, \beta]$ where $\alpha < \beta$. So the isolated points are exactly the successor ordinals and each limit ordinal is an accumulation point of the smaller ordinals.

Exercise 6.1. *Prove that any continuous $f : \omega_1 \to \mathbb{R}$ is eventually constant.*

Exercise 6.2. *Prove that any continuous $f : \mathbb{R} \to \omega_1$ has countable range.*

Exercise 6.3. *Show that any graph G of chromatic number at least $\mathfrak{c}^+$ must contain a copy of $H_{\omega,\mathfrak{c}^+}$.*

Exercise 6.4. *Prove Kőnig's theorem: any finitely branching, infinite tree must contain an infinite branch.*

Exercise 6.5. *Let $(e_\alpha)_{\alpha<\omega_1}$ be a coherent, 1-1 sequence and let $T = \{e_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}$. Prove that T is an Aronszajn-tree.*

Exercise 6.6. *Let $T \subset \omega^{<\omega_1}$ be an Aronszajn tree. Define a relation $<_\ell$ on T so that $s <_\ell t$ if $s(\xi) \supseteq t(\xi)$ or $s(\xi) < t(\xi)$ for the minimal ξ so that $s(\xi) \neq t(\xi)$. Prove that $<_\ell$ is a linear order.*

Problem 6.7. *Let T be the tree consisting of all closed subsets of a stationary set S. Show that T cannot be partitioned into countably many antichains.*

Problem 6.8. *Suppose that G is a graph on ω_1 and for any limit α, $\alpha \cap N(\alpha)$ is closed and discrete in the order topology of α. Prove that given any pairwise disjoint, uncountable family F of finite subsets of ω_1, there is a $a \neq b \in F$ so that there are no edges between a and b.*

Problem 6.9. *Let T be an Aronszajn subtree of $\omega^{<\omega_1}$. Show that there is no strictly increasing map from $(T, <_\ell)$ to $\mathbb{R}$ (see Exercise 6.6 for the definition of $<_\ell$).*

Problem 6.10. *Let $V = \{f : \alpha \to \omega \text{ injective}, \alpha < \omega_1\}$ and $fg \in E$ if $f \subset g$ or $g \subset f$. Prove that $G = (V, E)$ is uncountably chromatic.*