PROBLEM SET 5

DUE DATE: MAY 1, 2019

Exercise 5.1. *Show that any graph G of uncountable chromatic number contains a copy of K_{n,ω_1} for any $n<\omega$. The latter is the complete bipartite graph with one finite class of size n and another uncountable class.*

Exercise 5.2. *Fix a natural number k. Prove that any graph G of uncountable chromatic number contains a k-connected subgraph H i.e., H remains connected after the removal of $<k$ vertices.*

Exercise 5.3. *Prove that the rational distance graph on the plane has no copies of K_{2,ω_1} and so it must have countable chromatic number.*

Exercise 5.4. *Suppose that (X,τ) is a topological space with a point-countable base $\mathcal{B}$. That is, for any $x\in X$, $\{U\in\mathcal{B}: x\in U\}$ is countable. Let $(X,\tau),\mathcal{B}\in M\prec H(\theta)$. Prove that for any $y\in\overline{X\cap M}$, $\mathcal{B}\cap M$ contains a neighbourhood base for y.*

Exercise 5.5. *Suppose that X is a separable metric space and $Y\subset X$ is σ-discrete. Prove that Y is countable.*

Exercise 5.6. *Suppose that $2^{\aleph_0}=\aleph_1$ and X is an uncountable, separable metric space. Prove that X has a subset Y that is not G_δ.*

Problem 5.7. *Show that if a topological space X is σ-discrete and any subset is a G_δ then actually X is σ-closed discrete.*

Problem 5.8. *Suppose that some $X\subset 2^{\omega_1}$ satisfies the following: for any uncountable family $\{s_\xi:\xi<\omega_1\}$ of finite functions with pairwise disjoint domain there is a countable I so that $X\setminus\bigcup_{\xi\in I}[s_\xi]$ is countable.*[1] *Prove that any open cover of X has a countable subcover i.e., that X is Lindelöf.*

Problem 5.9. *Show that for any countable edge-colouring of the complete graph on ω_2, one can find an infinite monochromatic path.*

Problem 5.10. *Describe an explicit construction of finite triangle-free graphs with arbitrary large finite chromatic number.*[2]

Challenge 5.11. *Show that there is a countable subspace $X\subset 2^{\mathfrak{c}}$ which has no isolated points and any two non-empty dense subsets of X have non-empty intersection.*[3]

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Open Problem 5.12. *Is there a 'small' Dowker space i.e., one of size, character or weight ω_1?*

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[1] The notation $[s]$ stands for $\{f\in 2^{\omega_1}: s\subset f\}$.

[2] Hint: using a triangle-free graph H, try to build a larger G which is still triangle-free but has bigger chromatic number.

[3] Hint: construct $\{x_n:n<\omega\}\subset 2^{\mathfrak{c}}$ by defining $\{x_n\restriction\alpha:n<\omega\}$ by an induction on $\alpha<\mathfrak{c}$. What should we diagonalise in the construction?