PROBLEM SET 3

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Exercise 3.1. *Let d be a finite number. Prove that any infinite family of d-element sets contains an infinite Δ-system.*

Suppose that κ is an infinite cardinal. The basic open sets in the product topology on 2^κ are of the form $[\varepsilon] := \{f \in 2^\kappa : \varepsilon \subset f\}$ where ε is a finite partial function from κ to 2.

Exercise 3.2. *Let κ be an infinite cardinal. Show that there is no uncountable family of pairwise disjoint non-empty open subsets of 2^κ.*[1]

A topological space (X, τ) is called separable if it has a countable dense subset. (X, τ) is said to be Lindelöf if any open cover of X has a countable subcover.

Exercise 3.3. *Suppose that (X, τ) is a topological space and $(X, \tau) \in M \prec H(\theta)$. Prove the following claims.*

(1) If X is separable then $\overline{X \cap M} = X$.
(2) If X is Lindelöf and $\mathcal{U} \in M$ is an open cover of X then $M \cap \mathcal{U}$ covers X.

Exercise 3.4. *Suppose that $F : \omega_1 \to [\omega_1]^{<\omega}$. Show that there is a stationary $S \subset \omega_1$ so that $\{F(\xi) : \xi \in S\}$ is a Δ-system.*

Exercise 3.5. *Show that any family of $(2^{\aleph_0})^+$-many countably infinite sets contains a Δ-system of the same size.*

Problem 3.6. *Show that there is a family of countable sets $\mathcal{A}$ of size continuum so that $\mathcal{A}$ is totally ordered by the subset relation (i.e., for any $x \neq y \in A$ either $x \subset y$ or $y \subset x$).*

Problem 3.7. *Let k be finite and suppose that F is a family of sets each of finite size s. Show that if $|F| > s!\,(k-1)^s$ then F contains a Δ-system with at least k elements.*

Problem 3.8. *Suppose that $\mathcal{A}$ is a family of subsets of $\mathbb{R}$ and for any $a, b \in \mathcal{A}$, $a \cap b$ is finite. Prove that $\mathcal{A}$ has size at most continuum.*

Problem 3.9. *Prove that there is no strictly increasing sequence $(F_\xi)_{\xi<\omega_1}$ of closed subsets of $\mathbb{R}$.*

[1] In other words, this topology satisfies the countable chain condition.