PROBLEM SET 2

DUE DATE: MARCH 27, 2019

Exercise 2.1. *Prove that* $H(\aleph_1)$ *satisfies the statement 'all sets are countable.'*

Exercise 2.2. *Show that the transitive closure of any* $a \in H(\theta)$ *is again an element of* $H(\theta)$*. Show that* $[H(\theta)]^{<\theta} \subset H(\theta)$.

Exercise 2.3. *Suppose that* $(M_\alpha)_{\alpha<\mu}$ *is an increasing sequence of elementary submodels of some* $H(\theta)$*. Prove that* $M = \bigcup\{M_\alpha : \alpha < \mu\}$ *is also an elementary submodel of* $H(\theta)$.

Exercise 2.4. *Show that if* μ *is a cardinal which is an element and subset of* M *and* $X \in M$ *has size* μ *then* $X \subset M$ *as well.*

Exercise 2.5. *Show that any uncountable* $A \subset \mathbb{R}$ *with* $cf(|A|) > \omega$ *contains uncountably many complete accumulation points i.e.,* $x \in A$ *so that* $|U \cap A| = |A|$ *for any open neighbourhood* U *of* x.

Exercise 2.6. *Suppose that* $M \prec H(\theta)$ *is some elementary submodel and* $\kappa \in M$*. When is* $\kappa \cap M$ *an initial segment of* κ*?*

Problem 2.7. *Prove that, for large enough* θ*, for any* $X \subset H(\theta)$ *of size* $\mathfrak{c}$ *there is a countably closed elementary submodel* $M \prec H(\theta)$ *of size* $\mathfrak{c}$ *which contains* X.

Problem 2.8. *For any uncountable* $A \subset \mathbb{R}^n$*, there is an uncountable* $B \subset A$ *such that all distances between points in* B *are pairwise different.*

Problem 2.9. *Find a Borel function* $f : \mathbb{R} \to \mathbb{R}$ *with the property that* $f[I] = \mathbb{R}$ *for any non-empty open* $I \subset \mathbb{R}$.

Problem 2.10. *Show that* $\mathbb{R}$ *is the union of countably many sets* $(A_i)_{i<\omega}$ *so that none of the* A_i *contains a 3-element arithmetic progression.*

Problem 2.11. *Suppose that* G *is a graph and* $k < \omega$*. Assume that any finite subgraph of* G *has chromatic number at most* k*. Prove that* $\chi(G) \leq k$ *as well.*