PROBLEM SET 1

DUE DATE: MARCH 20, 2019

Exercise 1.1. *Starting with a single coin, you play a game with a simple automaton: at each step you insert a single coin to which the machine returns two coins.*

(1) *Show that an unassuming player might loose all his/her money in ω steps.*
(2) *Show that, with any strategy, the player will go bankrupt in countably many steps.*

Exercise 1.2. *Suppose that $A \subseteq \mathbb{C}$ is arbitrary. Show that there is an algebraically closed subfield $F \subseteq \mathbb{C}$ of size at most $|A|+\aleph_0$ which contains A.*

Exercise 1.3. *Suppose that $f_\alpha : \omega_1 \to \omega_1$ for $\alpha < \omega_1$. Show that there is a club $C \subset \omega_1$ such that for any $\beta \in C$ and $\alpha, \xi < \beta$, $f_\alpha(\xi) < \beta$ as well.*

Exercise 1.4. *Suppose that each line ℓ in $\mathbb{R}^2$ is assigned a natural number $m_\ell \geq 2$. Construct a set $A \subset \mathbb{R}^2$ which meets each line ℓ in exactly m_ℓ points.*

Problem 1.5. *Show that $\mathbb{R}^2$ has a well order $\prec$ so that for any y, the set $\{x \prec y : |y-x| \in \mathbb{Q}\}$ is finite. Why does it follow that the rational distance graph on $\mathbb{R}^2$ has countable chromatic number?*

Problem 1.6. *Prove that the family of all non-empty perfect subsets of $\mathbb{R}$ has chromatic number 2. In fact, show that there is a function $g : \mathbb{R} \to \mathbb{R}$ so that $g[C] = \mathbb{R}$ for any copy C of the Cantor set in $\mathbb{R}$.*

Problem 1.7. *Prove that any graph G of size at most continuum is* spatial *i.e., there is an injective $f : V(G) \to \mathbb{R}^3$ so that for any pair of edges $ab, cd \in E(G)$, the open line segments $(f(a), f(b))$ and $(f(c), f(d))$ are disjoint.*

Problem 1.8. *Can we cover $\mathbb{R}^2$ by disjoint circles? How about $\mathbb{R}^3$? Can you do it by unit circles only?*

Problem 1.9. *Show that $\mathbb{R}^+$ can be decomposed into two, disjoint sets both closed under addition.*

Challenge 1.10. *Show that the rational distance graph on $\mathbb{R}^3$ has countable chromatic number too.*

⋆

Open Problem 1.11 (Erdős)**.** *Is there a Borel set $A \subset \mathbb{R}^2$ which meets each line ℓ in exactly 2 points?*

Open Problem 1.12 (Ulam)**.** *Does there exist a dense set $S \subseteq \mathbb{R}^2$ so that all pairwise distances between points in S are rational?*

Open Problem 1.13. *Is there a Borel partition of $\mathbb{R}^3$ by unit circles?*