Partitioning bases of topological spaces

Dániel T. Soukup and Lajos Soukup
daniel.soukup@mail.utoronto.ca and soukup@renyi.hu

UNIVERSITY OF TORONTO

Introduction to the problem

By *space* we mean a topological space without isolated points.
Let us first introduce our **main problem** which is due to Barnabás Farkas:

Given a space X and a base $\mathbb{B}$ of X is there a partition of $\mathbb{B}$ into two bases?

Let's look at the literature!

- (Hewitt) Is there a partition of a space X into **disjoint dense sets**?
- (A. H. Stone) If a partial order $\mathbb{P}$ has **no maximal elements** then it admits a **partition to two cofinal sets**.
- (M. Elekes, T. Mátrai, L. Soukup) There is an **infinite fold cover** $\mathcal{A}$ of $\mathbb{R}$ with translates of a single compact set such that there are **no disjoint subcovers** of $\mathbb{R}$ in $\mathcal{A}$.
- (Lindgren, P. Nyikos) **Order properties of bases**, Noetherian bases, etc...

But not this particular question.... Hence we make the following

Definition. **A base $\mathbb{B}$ for a space X is resolvable iff it can be decomposed into two bases. A space X is base resolvable if every base of X is resolvable.**

Main Result I.

A space X is ***Lindelöf (compact)*** iff every cover of X has a countable (finite) subcover.

Theorem. *Every T_3 (locally) Lindelöf space is base resolvable.*

In particular, every **locally compact** or **locally countable** space is base resolvable!

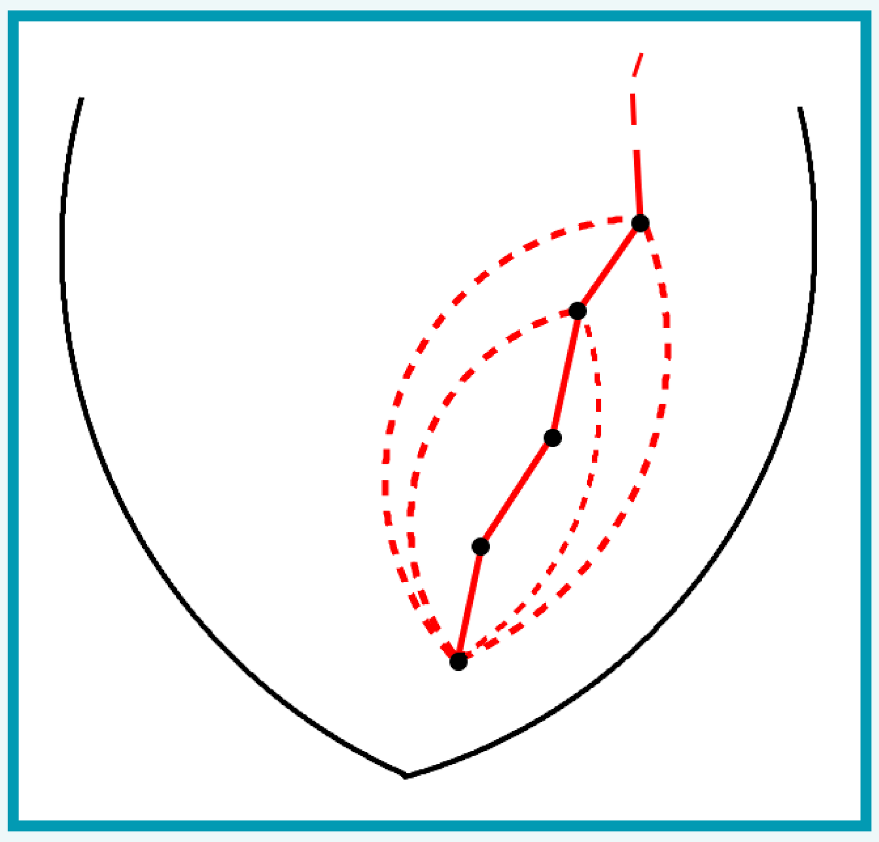
Is every space base resolvable?

First of all:

Proposition. *If a base for a T_1 topology is closed to finite unions then it is resolvable; in particular, every space admits many resolvable bases!*

Now, suppose that **$\mathbb{B}$ is a non resolvable base and lets look at the poset $\mathbb{P} = (\mathbb{B}, \supseteq)$**!

Observation. If we color $\mathbb{P}$ with two colors (red and blue, of course) then there is a a strictly increasing chain $(p_i)_{i\in\omega}$ in $\mathbb{P}$ and a color, say red, so that every $q \in \mathbb{P}$ is colored red if $p_0 \leq q \leq p_i$ for some $i \in \omega$.



We will denote this partition property by

$$\mathbb{P} \to (I_\omega)^1_2.$$

Theorem. *There is a locally finite poset $\mathbb{P}$ of size ω_1 which satisfies $\mathbb{P} \to (I_\omega)^1_2$.*

Main Result II.

Using a partial order $\mathbb{P}$ with $\mathbb{P} \to (I_\omega)^1_2$ we can prove the following

Theorem. *There is a (T_0) space X with a point countable, non resolvable base $\mathbb{B}$.*

What is the main idea?

- the points of X are $x = (p_i)_{i\in\omega}$ **increasing chains** in $\mathbb{P}$,
- let
  $$U_q = \{(p_i)_{i\in\omega} \in X : \exists i \in \omega (q \leq p_i)\}$$
  for each $q \in \mathbb{P}$,
- $\mathbb{B} = \{U_q : q \in \mathbb{P}\}$ will form a **base for a topology** which is **not resolvable** by the partition property.

Not even Hausdorff?? That's quite unsatisfactory...

Theorem (L. Soukup). *Consistently, there is a 0-dimensional, first countable and Hausdorff space X which has a non resolvable base.*

Let's use the previous idea:

- introduce a poset $\mathbb{P}$ by **forcing with finite conditions**,
- introduce the increasing chains (points of the space) generically as well,
- start calculating like hell and hope for the best!

Observations

Firstly,

1. **Every base can be partitioned to a cover and a base.**

Applying Stone's result gives that

2. **Every neighborhood base of a point can be partitioned into two neighborhood bases.**
3. Every π-base can be partitioned to two π-bases.

Recall that a **π-base** of a space X is a family of nonempty open sets $\mathcal{U}$ such that for every non empty open $V \subseteq X$ there is $U \in \mathcal{U}$ with $U \subseteq V$.

Metrizable spaces

Let's see some proof!

Proposition. *Every metrizable space is base resolvable.*

Proof. Fix a base $\mathbb{B}$ for decomposition and ...
...find another base $\mathbb{B}_\sigma$ with the property

$(*)$ $\mathbb{B}_\sigma = \cup\{\mathbb{B}_n : n \in \omega\}$ where each $\mathbb{B}_n$ is a disjoint family;

this can be done by metrizability!
Now select pairwise disjoint $\mathcal{U}_B, \mathcal{V}_B \subseteq \mathbb{B}$ for each $B \in \mathbb{B}_0$ such that

$$\cup\mathcal{U}_B = \cup\mathcal{V}_B = B$$

and $\mathbb{B} \setminus (\mathcal{U}_B \cup \mathcal{V}_B)$ is still a base; note that $(*)$ ensures that the rest of $\mathbb{B}$ is still a base!
Repeat for each $\mathbb{B}_n$ inductively and $\mathcal{U} = \cup\{\mathcal{U}_B : B \in \mathbb{B}_\sigma\}$ and $\mathcal{V} = \cup\{\mathcal{V}_B : B \in \mathbb{B}_\sigma\}$ will be disjoint bases! □

Open problems

- Is every **linearly ordered** space base resolvable?
- Is every T_3 (hereditarily) **separable** space base resolvable?
- Is every **homogeneous** space base resolvable?
- Is every **power of $\mathbb{R}$** base resolvable?

Thanks

Acknowledgments. The first author would like to thank his advisor, Bill Weiss for the long hours of discussions. We thank Barnabás Farkas for the nice problem and for the help with this poster!

Access to paper:
http://www.math.toronto.edu/∼ dsoukup/