Monochromatic path decompositions

Dániel T. Soukup

University of Toronto

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The origins

Theorem (R. Rado, 1978)

*Given an **r-edge colouring of the complete graph** on $\mathbb{N}$ with r finite, we can **partition** the vertices into r-many disjoint **monochromatic paths** of different colour.*

P. Erdős on Richard Rado:

"I was good at discovering perhaps difficult and interesting special cases, and Richard was good at generalising them and putting them in their proper perspective."

Developements on the finite case

General problem (Gyárfás): given an r-edge colouring of K_n is there a cover by (vertex disjoint) monochromatic paths (of differenct colour)?

Suppose that r **is small**:

1. ("easy") Every 2-edge coloured K_n can be partitioned into 2 monochromatic paths of different colour.
2. (K. Heinrich, ??) There are r-edge coloured copies of K_n for $r \geq 3$ so that there is no partition into r paths of different colour.
3. (A. Pokrovskiy, 2013) Every 3-edge coloured K_n can be partitioned into 3 monochromatic paths.

Completely open: $r = 4$ or larger.

Developements on the finite case

General problem (Gyárfás): given an r-edge colouring of K_n is there a cover by (vertex disjoint) monochromatic paths (of differenct colour)?

For arbitrary number of colours:

1. (Gyárfás, 1989) Every r-edge coloured K_n is **covered by** $\leq C \cdot r^4$ monochromatic paths (for some small constant C).
2. (Gyárfás et al., 1998) Every r-edge coloured copy of K_n can be **partitioned into** $\approx 100r\log(r)$ monochromatic cycles.

Significant work done on **monochromatic cycle partitions**; **Lehel's conjecture** and **Erdős-Gyárfás-Pyber conjecture**.

A proof of Rado's theorem

With a hint of set theory...

$\mathbb{N}$ admits **non-trivial 0-1 valued measures**, meaning

- $\mathbb{N}$ has measure 1,
- if A has measure 1 and $A \subset B$ then B has measure 1,
- if A, B have measure 1 then $A \cap B$ has measure 1 and is infinite,
- either A or $\mathbb{N} \setminus A$ has measure 1.

Remark: this is usually referred to as a **non principal ultrafilter**.

A proof of Rado's theorem

With a hint of set theory...

Proof.

- suppose that $c : [\mathbb{N}]^2 \to r$ and fix a **measure** as above,
- let $A_i = \{v \in \mathbb{N} : N(v, i)$ **has measure** $1\}$ for $i < r$,
- for every $v \in \mathbb{N}$ there is $i = i_v < r$ so that $N(v, i)$ has measure 1 so $\{A_i : i < r\}$ **partitions** $\mathbb{N}$,
- any $v, w \in A_i$ is **connected by infinitely many paths of colour** i,
- construct r disjoint monochromatic paths $\{P_i : i < r\}$ **inductively covering more and more points** of each A_i with P_i.

□

Theorem (D.S.)

*Suppose that G is the countably **infinite complete** balanced **bipartite** graph.*

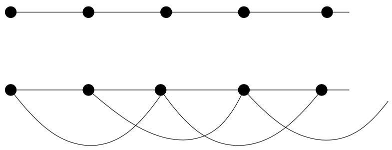
1. *if c is an r-edge colouring of G (with $r \in \mathbb{N}$) then the vertices can be **partitioned into** $\leq 2r - 1$ **monochromatic paths**,*
2. *there is an r-edge colouring of G (for each $r \in \mathbb{N}$) such that the vertices cannot be partitioned into $< 2r - 1$ monochromatic paths.*

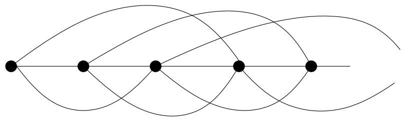
Note: the ultrafilter trick works more generally, e.g. on the **Random-graph**.

Definition

Suppose that G is a graph and $k \in \omega$. The ***kth power of G*** *is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a* ***finite path of length $\leq k$ from v to w***.

What is a **power of a path**?





Generalizations on $\mathbb{N}$

A **kth-power of a path** is $\{x_i : i < n\}$ so that x_i, x_j is an edge if $|i - j| \leq k$.

Theorem

Fix natural numbers k, r and an r-edge colouring c of the complete graph on $\mathbb{N}$. Then the vertices can be ***partitioned into $\leq r^{kr+1}$ infinite monochromatic kth powers of paths*** *and a finite set.*

Jointly with M. Elekes, L. Soukup and Z. Szentmiklóssy at Rényi Institute.

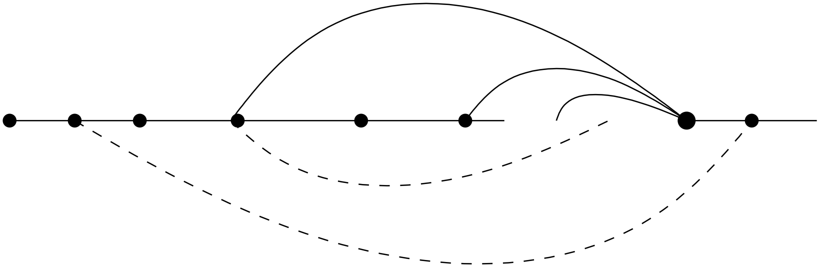
For $k = r = 2$ we actually have a **partition into 5** monochromatic second powers of paths.

Infinite paths of arbitrary length

Definition (Rado, 1978)

For a graph $P = (V, E)$, we say that P is a ***path*** iff there is a **well ordering $\prec$ on P** such that any two points $v, w \in V$ are connected by a $\prec$-monotone finite path.

How to imagine paths longer than the type of $\mathbb{N}$? **Type $\mathbb{N}+1$?**



Large infinite complete graphs

A **path** is a graph P with w.o. $\prec$ so that any two points are connected by a finite $\prec$-monotone path.

Theorem (D.S.)

*If G is an **infinite complete graph and** $r \in \mathbb{N}$ then for every r-edge colouring of G we can **partition** the vertices **into finitely many monochromatic paths**.*

Is it true that the number of paths needed only **depends on the number of colours**?

The number of colours and the **size** of G?

More problems and generalizations

A ***generalized path*** is a graph P with ordering $<$ so that any two points are connected by a finite $<$-monotone path.

Eg: we consider paths of **type** $\mathbb{Q}$, type $\mathbb{R}$, etc.

Problem

*Suppose that the **countably infinite complete graph** is coloured by finitely many colours. Is it true that the vertices can be **partitioned into/covered by** finitely many monochromatic paths of **type** $\mathbb{Q}$?*

Thank you...

for your attention so early in the morning!

Any questions?