Problems on uncountable graphs
- the morning show -

Dániel T. Soukup

http://renyi.hu/∼dsoukup/

Introduction

Goal: present **a selection of chromatic number problems** on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...
- historical and personal reasons,
- minimal structure
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

Introduction

Goal: present **a selection of chromatic number problems** on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...
- historical and personal reasons,
- minimal structure
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

Introduction

Goal: present **a selection of chromatic number problems** on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...
- **historical** and **personal** reasons,
- minimal structure
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

Introduction

Goal: present **a selection of chromatic number problems** on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...
- **historical** and **personal** reasons,
- **minimal structure**
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

Introduction

Goal: present a selection of chromatic number problems on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...

- historical and personal reasons,
- minimal structure
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

Introduction

Goal: present a selection of chromatic number problems on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...
- historical and personal reasons,
- minimal structure
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

Introduction

Goal: present a selection of chromatic number problems on uncountable graphs.

- some very good surveys: P. Komjáth, S. Todorcevic...
- historical and personal reasons,
- minimal structure
  - easily accessible problems,
  - great for understanding the limitations of certain techniques.

I received kind support and advice from several people: I. Juhász, P. Komjáth, L. Soukup, J. Steprans, S. Todorcevic, the Cambridge Phil. Soc.

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*

How does **large chromatic number** affect the **subgraph structure**?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*

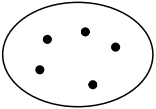
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*



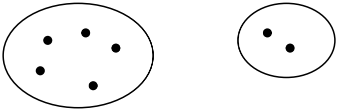
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow$ AC

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*



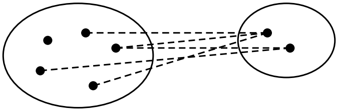
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

What is the chromatic number?

Definition

The ***chromatic number*** *of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that* ***the vertices of G can be covered by κ many independent sets****.*



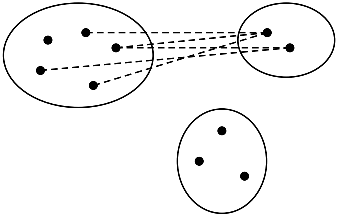
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*



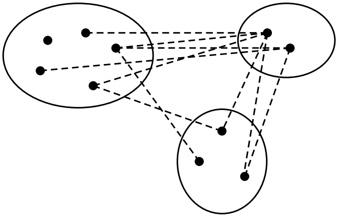
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow$ AC

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*



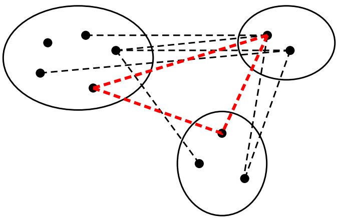
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow \mathrm{AC}$

What is the chromatic number?

Definition

The ***chromatic number*** *of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that* ***the vertices of G can be covered by κ many independent sets****.*



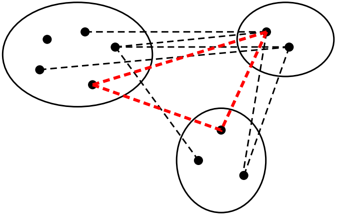
How does large chromatic number affect the subgraph structure?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*



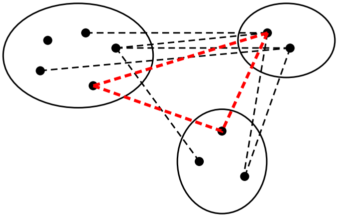
How does **large chromatic number** affect the **subgraph structure**?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

What is the chromatic number?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*



How does **large chromatic number** affect the **subgraph structure**?

[Galvin, Komjáth 1991]:
$\forall G\ \exists Chr(G) \Leftrightarrow AC$

The first results

- **Tutte, 1954:** There are $\triangle$-free graphs of arbitrary large finite chromatic number.
- **Erdős, 1959:** There are graphs with arbitrary large girth and arbitrary large finite chromatic number.

- **Tutte, 1954:** There are △-free graphs of **arbitrary large finite chromatic number**.

- **Erdős, 1959:** There are graphs with **arbitrary large girth** and **arbitrary large finite chromatic number**.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

- **Tutte, 1954:** There are △-**free** graphs of **arbitrary large finite chromatic number**.
- **Erdős, 1959:** There are graphs with **arbitrary large girth** and **arbitrary large finite chromatic number**.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

Obligatory subgraphs

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are **△-free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

In particular, **any finite bipartite graph embeds** into G.

Obligatory subgraphs

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are $\triangle$**-free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

In particular, **any finite bipartite graph embeds** into G.

Obligatory subgraphs

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are **$\triangle$-free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:**
If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

In particular, **any finite bipartite graph embeds** into G.

Obligatory subgraphs

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are **△-free** graphs with size and **chromatic number κ** for each infinite κ.

- **Erdős-Hajnal, 1966:**
  If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

In particular, **any finite bipartite graph embeds** into G.

Further finite obligatory subgraphs...

What finite graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Hajnal, 1966:** For any $n \in \mathbb{N}$ there is a graph G with $Chr(G) = \omega_1$ such that G does **not contain odd cycles of length** $< n$.
- **Erdős et al, Thomassen 1983:** If $Chr(G) > \omega$ then there is an $n \in \omega$ such that any **odd cycle of length bigger than** n **embeds into** G.

The finite obligatory graphs are exactly the finite bipartite graphs.

Further finite obligatory subgraphs...

What finite graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Hajnal, 1966:** For any $n \in \mathbb{N}$ there is a graph G with $Chr(G) = \omega_1$ such that G does **not contain odd cycles of length** $< n$.
- **Erdős et al, Thomassen 1983:** If $Chr(G) > \omega$ then there is an $n \in \omega$ such that any **odd cycle of length bigger than** n **embeds into** G.

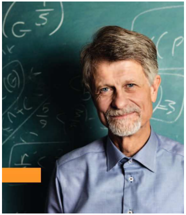
The finite obligatory graphs are exactly the finite bipartite graphs.

Further finite obligatory subgraphs...

What finite graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Hajnal, 1966:** For any $n \in \mathbb{N}$ there is a graph G with $Chr(G) = \omega_1$ such that G does **not contain odd cycles of length** $< n$.
- **Erdős et al, Thomassen 1983:** If $Chr(G) > \omega$ then there is an $n \in \omega$ such that any **odd cycle of length bigger than** n **embeds into** G.

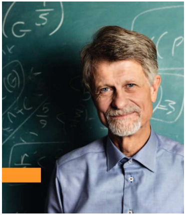
The finite obligatory graphs are exactly the finite bipartite graphs.



Further finite obligatory subgraphs...

What finite graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Hajnal, 1966:** For any $n \in \mathbb{N}$ there is a graph G with $Chr(G) = \omega_1$ such that G does **not contain odd cycles of length** $< n$.
- **Erdős et al, Thomassen 1983:** If $Chr(G) > \omega$ then there is an $n \in \omega$ such that any **odd cycle of length bigger than** n **embeds into** G.



The finite obligatory graphs are exactly the finite bipartite graphs.

Countable obligatory subgraphs

Can we classify countable obligatory subgraphs?

- **Hajnal-Komjáth, 1984:** $H_{\omega,\omega+1}$ embeds into G if $Chr(G) > \omega$ but $K_{\omega,\omega}$ **can be avoided**.

Problem

Is there a **universal countable obligatory graph** H^* for graphs G with $Chr(G) = \omega_1$?

There is a universal countable obligatory graph for graphs G with $Col(G) = \omega_1$.

Countable obligatory subgraphs

Can we classify countable obligatory subgraphs?

- **Hajnal-Komjáth, 1984:** $H_{\omega,\omega+1}$ embeds into G if $Chr(G) > \omega$ but $K_{\omega,\omega}$ **can be avoided**.

Problem

Is there a **universal countable obligatory graph** H^* for graphs G with $Chr(G) = \omega_1$?

There is a universal countable obligatory graph for graphs G with $Col(G) = \omega_1$.

Countable obligatory subgraphs

Can we classify countable obligatory subgraphs?

- **Hajnal-Komjáth, 1984:** $H_{\omega,\omega+1}$ embeds into G if $Chr(G) > \omega$ but $K_{\omega,\omega}$ **can be avoided**.

Problem

Is there a **universal countable obligatory graph** H^* for graphs G with $Chr(G) = \omega_1$?

There is a universal countable obligatory graph for graphs G with $Col(G) = \omega_1$.

Countable obligatory subgraphs

Can we classify countable obligatory subgraphs?

- **Hajnal-Komjáth, 1984:** $H_{\omega,\omega+1}$ embeds into G if $Chr(G) > \omega$ but $K_{\omega,\omega}$ **can be avoided**.

Problem

Is there a **universal countable obligatory graph** H^* for graphs G with $Chr(G) = \omega_1$?

There is a universal countable obligatory graph for graphs G with $Col(G) = \omega_1$.

Chromatic number and connectivity

A set A in graph is **infinitely connected** iff A is infinite and $A \setminus F$ is connected for all finite $F \subseteq A$.

[Erdős-Hajnal, 1966, 1985...] Does every graph G with uncountable chromatic number **contain infinitely connected subgraphs** with uncountable chromatic number?

- **Komjáth, 1988:** It is **independent of ZFC** if every graph G with $|G| = Chr(G) = \omega_1$ contains an infinitely connected uncountably chromatic subgraph.

Chromatic number and connectivity

A set A in graph is **infinitely connected** iff A is infinite and $A \setminus F$ is connected for all finite $F \subseteq A$.

[Erdős-Hajnal, 1966, 1985...] Does every graph G with uncountable chromatic number **contain infinitely connected subgraphs** with uncountable chromatic number?

- **Komjáth, 1988:** It is **independent of ZFC** if every graph G with $|G| = Chr(G) = \omega_1$ contains an infinitely connected uncountably chromatic subgraph.

Chromatic number and connectivity

A set A in graph is **infinitely connected** iff A is infinite and $A \setminus F$ is connected for all finite $F \subseteq A$.

[Erdős-Hajnal, 1966, 1985...] Does every graph G with uncountable chromatic number **contain infinitely connected subgraphs** with uncountable chromatic number?

- **Komjáth, 1988:** It is **independent of ZFC** if every graph G with $|G| = Chr(G) = \omega_1$ contains an infinitely connected uncountably chromatic subgraph.

Chromatic number and connectivity

Erdős-Hajnal, 1985: What about graphs with $Chr(G) = \omega_1$ (no size restriction)?

Theorem (D.S. 2015)

*There is (in ZFC) a graph of **chromatic number** ω_1 and size continuum **without uncountable infinitely connected subgraphs**.*

- the chromatic number and cardinality are **best possible**,
- **sparse triangle-free graphs** with uncountable chromatic number can be produced with the same machinery.

Chromatic number and connectivity

Erdős-Hajnal, 1985: What about graphs with $Chr(G) = \omega_1$ (no size restriction)?

Theorem (D.S. 2015)

*There is (in ZFC) a graph of **chromatic number ω_1** and size continuum **without uncountable infinitely connected subgraphs**.*

- the chromatic number and cardinality are best possible,
- sparse triangle-free graphs with uncountable chromatic number can be produced with the same machinery.

Chromatic number and connectivity

Erdős-Hajnal, 1985: What about graphs with $Chr(G) = \omega_1$ (no size restriction)?

Theorem (D.S. 2015)

There is (in ZFC) a graph of ***chromatic number ω_1*** *and size continuum* ***without uncountable infinitely connected subgraphs****.*

- the chromatic number and cardinality are **best possible**,
- sparse triangle-free graphs with uncountable chromatic number can be produced with the same machinery.

Chromatic number and connectivity

Erdős-Hajnal, 1985: What about graphs with $Chr(G) = \omega_1$ (no size restriction)?

Theorem (D.S. 2015)

*There is (in ZFC) a graph of **chromatic number** ω_1 and size continuum **without uncountable infinitely connected subgraphs**.*

- the chromatic number and cardinality are **best possible**,
- **sparse triangle-free graphs** with uncountable chromatic number can be produced with the same machinery.

A few words about the proof

- consider the comparability graph of a non-special tree without uncountable chains.

- thin out the edges to have no uncountable infinitely connected subgraph:
  - use a ladder system on the tree;

- how to make sure that the chromatic number is still large?
  - a diagonalization of length continuum through countable elementary submodels.

A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**

A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

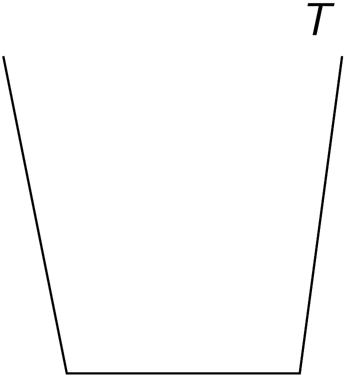
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**

A few words about the proof

- consider the comparability graph of a non-special tree without uncountable chains.

- thin out the edges to have no uncountable infinitely connected subgraph:
  - use a ladder system on the tree;
- how to make sure that the chromatic number is still large?
  - a diagonalization of length continuum through countable elementary submodels.

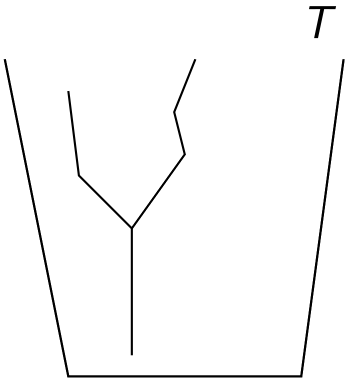


A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

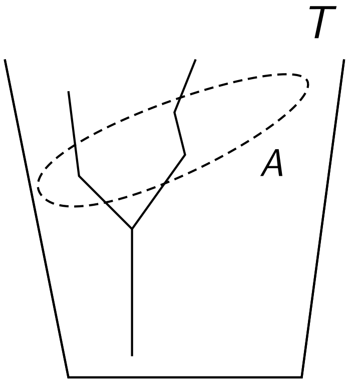
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels**.



A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**

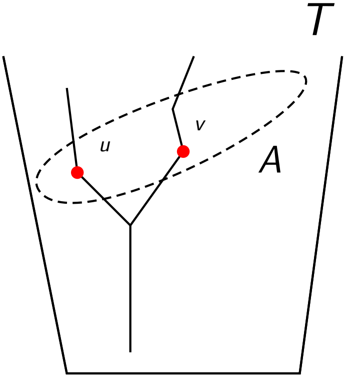


A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**

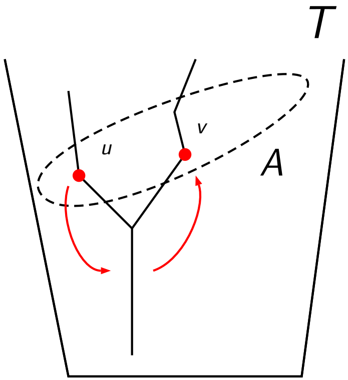


A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**

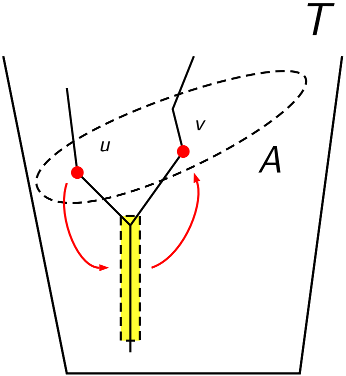


A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

- thin out the edges to have no uncountable infinitely connected subgraph:
  - use a ladder system on the tree;

- how to make sure that the chromatic number is still large?
  - a diagonalization of length continuum through countable elementary submodels.

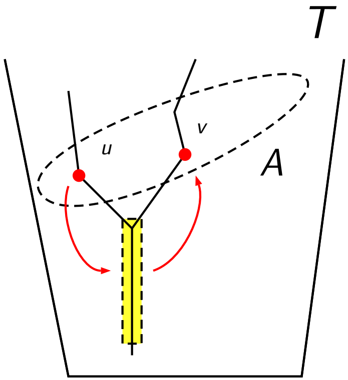


A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

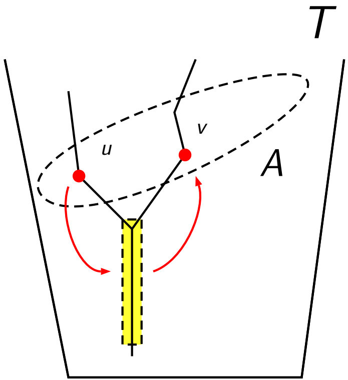
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**



A few words about the proof

- consider the comparability graph of a non-special tree without uncountable chains.
- thin out the edges to have no uncountable infinitely connected subgraph:
  - use a ladder system on the tree;
- how to make sure that the chromatic number is still large?
  - a diagonalization of length continuum through countable elementary submodels.

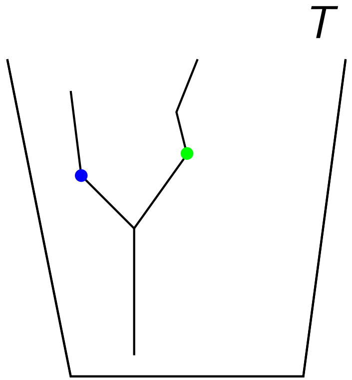


A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

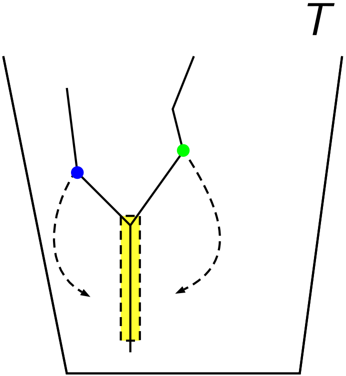
- **thin out the edges** to have no uncountable infinitely connected subgraph:
    - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
    - a **diagonalization** of length continuum through **countable elementary submodels**.



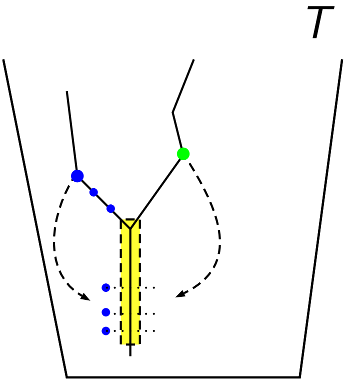
A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**



A few words about the proof

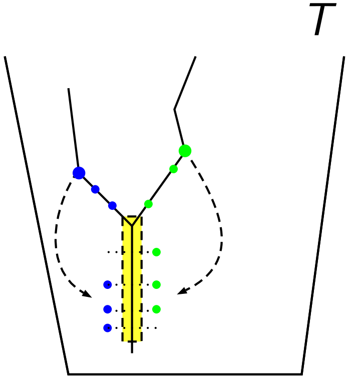
- consider the **comparability graph** of a **non-special tree without uncountable chains**.
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;



- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels**.

A few words about the proof

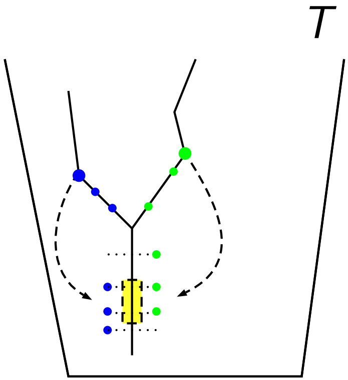
- consider the **comparability graph** of a **non-special tree without uncountable chains**.
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels**.



A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;



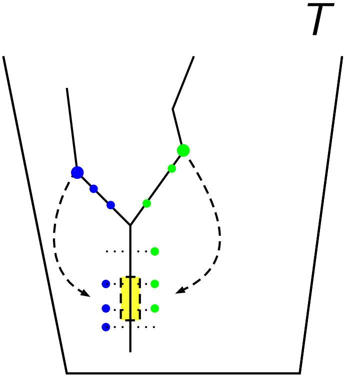
- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**

A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.

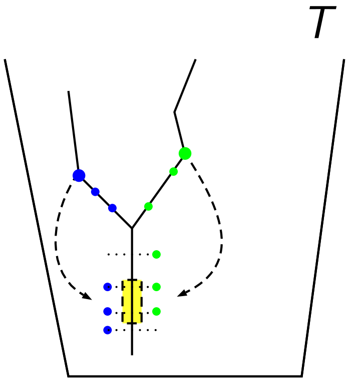
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;

- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**



A few words about the proof

- consider the **comparability graph** of a **non-special tree without uncountable chains**.
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **diagonalization** of length continuum through **countable elementary submodels.**



Chromatic number and connectivity

What is left to do?

Problem [Erdős]

Is there a graph G with chromatic number ω_1 **without non empty infinitely connected subgraphs**?

My example is **full of countably infinite complete subgraphs**.

Can we find a **triangle-free subgraph** at least?

Chromatic number and connectivity

What is left to do?

Problem [Erdős]

Is there a graph G with chromatic number ω_1 **without non empty infinitely connected subgraphs**?

My example is **full of countably infinite complete subgraphs**.

Can we find a **triangle-free subgraph** at least?

Problem [Erdős]

Is there a graph G with chromatic number ω_1 **without non empty infinitely connected subgraphs**?

My example is **full of countably infinite complete subgraphs**.

Can we find a **triangle-free subgraph** at least?

More on triangle-free subgraphs

How common are the triangle-free graphs with large chromatic number?

Conjecture [Erdős]

Every graph G **with** $Chr(G) = \omega_1$ contains
a subgraph H with $Chr(H) = \omega_1$ **without triangles**.

Komjáth, Shelah 1988: consistently no.

- There is G with size and chromatic number ω_1 such that **any subgraph** H with $Chr(H) = \omega_1$ **contains a copy of** K_ω.
- There is a K_4**-free** G with size and chromatic number ω_1 such that any subgraph H with $Chr(H) = \omega_1$ **contains a triangle**.

More on triangle-free subgraphs

How common are the triangle-free graphs with large chromatic number?

Conjecture [Erdős]

Every graph G with $Chr(G) = \omega_1$ contains
a subgraph H with $Chr(H) = \omega_1$ **without triangles**.

Komjáth, Shelah 1988: consistently no.

- There is G with size and chromatic number ω_1 such that **any subgraph H** with $Chr(H) = \omega_1$ **contains a copy of K_ω**.
- There is a K_4-free G with size and chromatic number ω_1 such that any subgraph H with $Chr(H) = \omega_1$ **contains a triangle**.

More on triangle-free subgraphs

How common are the triangle-free graphs with large chromatic number?

Conjecture [Erdős]

Every graph G **with** $Chr(G) = \omega_1$ contains
a subgraph H with $Chr(H) = \omega_1$ **without triangles**.

Komjáth, Shelah 1988: consistently no.

- There is G with size and chromatic number ω_1 such that **any subgraph** H with $Chr(H) = \omega_1$ **contains a copy of** K_ω.
- There is a K_4-free G with size and chromatic number ω_1 such that any subgraph H with $Chr(H) = \omega_1$ **contains a triangle**.

More on triangle-free subgraphs

How common are the triangle-free graphs with large chromatic number?

Conjecture [Erdős]

Every graph G **with** $Chr(G) = \omega_1$ contains
a subgraph H with $Chr(H) = \omega_1$ **without triangles**.

Komjáth, Shelah 1988: consistently no.

- There is G with size and chromatic number ω_1 such that **any subgraph** H with $Chr(H) = \omega_1$ **contains a copy of** K_ω.
- There is a K_4-free G with size and chromatic number ω_1 such that any subgraph H with $Chr(H) = \omega_1$ contains a triangle.

More on triangle-free subgraphs

How common are the triangle-free graphs with large chromatic number?

Conjecture [Erdős]

Every graph G with $Chr(G) = \omega_1$ contains **a subgraph H** with $Chr(H) = \omega_1$ **without triangles**.

Komjáth, Shelah 1988: consistently no.

- There is G with size and chromatic number ω_1 such that **any subgraph H** with $Chr(H) = \omega_1$ **contains a copy of K_ω**.
- There is a **K_4-free** G with size and chromatic number ω_1 such that any subgraph H with $Chr(H) = \omega_1$ **contains a triangle**.

More on triangle-free subgraphs

Both examples are introduced by classical iterated forcings:

- *Step 1*: force a generic graph (either with finite or countable approximations),
- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see more constructive examples, e.g. from $\diamondsuit$ and in ZFC???

More on triangle-free subgraphs

Both examples are introduced by **classical iterated forcings**:

- *Step 1*: force **a generic graph** (either with finite or countable approximations),
- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see **more constructive examples**, e.g. from $\diamondsuit$ and in **ZFC???**

More on triangle-free subgraphs

Both examples are introduced by **classical iterated forcings**:

- *Step 1*: force **a generic graph** (either with finite or countable approximations),

- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see **more constructive examples**, e.g. from $\diamondsuit$ and in **ZFC???**

More on triangle-free subgraphs

Both examples are introduced by **classical iterated forcings**:

- *Step 1*: force **a generic graph** (either with finite or countable approximations),
- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see **more constructive examples**, e.g. from $\diamondsuit$ and in **ZFC???**

More on triangle-free subgraphs

Both examples are introduced by **classical iterated forcings**:

- *Step 1*: force **a generic graph** (either with finite or countable approximations),
- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see **more constructive examples**, e.g. from $\diamondsuit$ and in **ZFC???**

More on triangle-free subgraphs

Both examples are introduced by **classical iterated forcings**:

- *Step 1*: force **a generic graph** (either with finite or countable approximations),

- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see more constructive examples, e.g. from $\diamondsuit$ and in ZFC???

More on triangle-free subgraphs

Both examples are introduced by classical iterated forcings:

- *Step 1*: force a generic graph (either with finite or countable approximations),
- *Step 2*: do an iteration to **introduce ω-partitions of triangle-free subgraphs into independent sets** (FSI or CSI respectively),
- *Step 3*: show that this works (also note the value of $\mathfrak{c}$)...

What **structural requirements** imply the lack of large Δ-free subgraphs??

It would be interesting to see more constructive examples, e.g. from $\diamondsuit$ and in **ZFC???**

Finding common subgraphs

Any two graphs G_0, G_1 with uncountable chromatic number contain a common 3-chromatic subgraph.

Problem [Erdős]

Does every two graphs G_0, G_1 with uncountable chromatic number contain **a common 4-chromatic subgraph?** Is there a common ω-chromatic subgraph?

Finding common subgraphs

Any two graphs G_0, G_1 with uncountable chromatic number contain a common 3-chromatic subgraph.

Problem [Erdős]

Does every two graphs G_0, G_1 with uncountable chromatic number contain **a common 4-chromatic subgraph?** Is there a common ω-chromatic subgraph?

Finding common subgraphs

Any two graphs G_0, G_1 with uncountable chromatic number contain a common 3-chromatic subgraph.

Problem [Erdős]

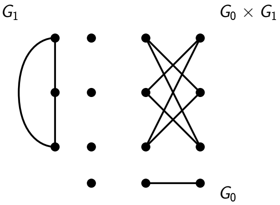
Does every two graphs G_0, G_1 with uncountable chromatic number contain **a common 4-chromatic subgraph?** Is there a common ω-chromatic subgraph?

Common subgraphs and products

The **product** $G_0 \times G_1$ of $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ has vertices $V_0 \times V_1$ and edges $\{(x,y),(x',y')\}$ where $\{x,x'\} \in E_0$ and $\{y,y'\} \in E_1$.

- If $H \hookrightarrow G_0, G_1$ then $H \hookrightarrow G_0 \times G_1$ as well.
- There are graphs G_0 and G_1 with **chromatic number** ω_1 such that $Chr(G_0 \times G_1) \leq \omega$ (see **Hedetniemi's** conjecture).
- These graphs have **no common** ω_1**-chromatic subgraph**.

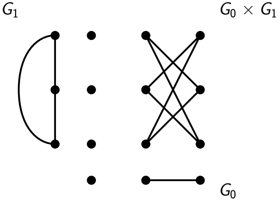
The **product** $G_0 \times G_1$ of $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ has vertices $V_0 \times V_1$ and edges $\{(x, y), (x', y')\}$ where $\{x, x'\} \in E_0$ and $\{y, y'\} \in E_1$.



- If $H \hookrightarrow G_0, G_1$ then $H \hookrightarrow G_0 \times G_1$ as well.
- There are graphs G_0 and G_1 with **chromatic number** ω_1 such that $Chr(G_0 \times G_1) \leq \omega$ (see **Hedetniemi's** conjecture).
- These graphs have **no common ω_1-chromatic subgraph**.

Common subgraphs and products

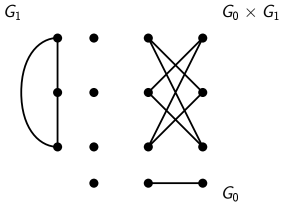
The **product** $G_0 \times G_1$ of $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ has vertices $V_0 \times V_1$ and edges $\{(x,y),(x',y')\}$ where $\{x,x'\} \in E_0$ and $\{y,y'\} \in E_1$.



- If $H \hookrightarrow G_0, G_1$ then $H \hookrightarrow G_0 \times G_1$ as well.
- There are graphs G_0 and G_1 with **chromatic number** ω_1 such that $Chr(G_0 \times G_1) \leq \omega$ (see **Hedetniemi's** conjecture).
- These graphs have **no common ω_1-chromatic subgraph**.

Common subgraphs and products

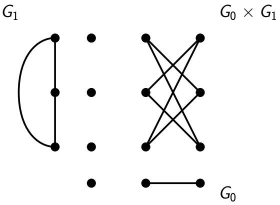
The **product** $G_0 \times G_1$ of $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ has vertices $V_0 \times V_1$ and edges $\{(x,y),(x',y')\}$ where $\{x,x'\} \in E_0$ and $\{y,y'\} \in E_1$.



- If $H \hookrightarrow G_0, G_1$ then $H \hookrightarrow G_0 \times G_1$ as well.
- There are graphs G_0 and G_1 with **chromatic number** ω_1 such that $Chr(G_0 \times G_1) \leq \omega$ (see **Hedetniemi's** conjecture).
- These graphs have **no common ω_1-chromatic subgraph**.

Common subgraphs and products

The **product** $G_0 \times G_1$ of $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ has vertices $V_0 \times V_1$ and edges $\{(x, y), (x', y')\}$ where $\{x, x'\} \in E_0$ and $\{y, y'\} \in E_1$.



- If $H \hookrightarrow G_0, G_1$ then $H \hookrightarrow G_0 \times G_1$ as well.
- There are graphs G_0 and G_1 with **chromatic number** ω_1 such that $Chr(G_0 \times G_1) \leq \omega$ (see **Hedetniemi's** conjecture).
- These graphs have **no common ω_1-chromatic subgraph**.

More on common subgraphs and products

Problem [Hajnal, L. Soukup]

Is there a graph G such that $Chr(G) = Chr(G^C) = \omega_1$ but $Chr(G \times G^C) \leq \omega$?

Is there always **a common** ω_1**-chromatic subgraph of** G **and** G^C provided $Chr(G) = Chr(G^C) = \omega_1$?

Problem

Construct G_0 and G_1 such that $Chr(G_0 \times G_1) = \omega_1$ but G_0 and G_1 has **no common** ω_1**-chromatic subgraphs**.

More on common subgraphs and products

Problem [Hajnal, L. Soukup]

Is there a graph G such that $Chr(G) = Chr(G^C) = \omega_1$ but $Chr(G \times G^C) \leq \omega$?

Is there always **a common ω_1-chromatic subgraph of G and G^C** provided $Chr(G) = Chr(G^C) = \omega_1$?

Problem

Construct G_0 and G_1 such that $Chr(G_0 \times G_1) = \omega_1$ but G_0 and G_1 has **no common ω_1-chromatic subgraphs**.

More on common subgraphs and products

Problem [Hajnal, L. Soukup]

Is there a graph G such that $Chr(G) = Chr(G^C) = \omega_1$ but $Chr(G \times G^C) \leq \omega$?

Is there always **a common ω_1-chromatic subgraph of G and G^C** provided $Chr(G) = Chr(G^C) = \omega_1$?

Problem

Construct G_0 and G_1 such that $Chr(G_0 \times G_1) = \omega_1$ but G_0 and G_1 has **no common ω_1-chromatic subgraphs**.

More on common subgraphs and products

Problem [Hajnal, L. Soukup]

Is there a graph G such that $Chr(G) = Chr(G^C) = \omega_1$ but $Chr(G \times G^C) \leq \omega$?

Is there always **a common ω_1-chromatic subgraph of G and G^C** provided $Chr(G) = Chr(G^C) = \omega_1$?

Problem

Construct G_0 and G_1 such that $Chr(G_0 \times G_1) = \omega_1$ but G_0 and G_1 has **no common ω_1-chromatic subgraphs**.

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.

G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C.
G is **almost smooth** iff $G[W]$ is isomorphic to G whenever $|V \setminus W| < |V|$.

Problem [Kierstead, Nyikos 1989]

Is there a non trivial almost smooth (hyper)graph G?

Hajnal, Nagy, L. Soukup 1990: consistently yes.

- under CH, or
- consistently with MA.

Is there a non trivial almost smooth (hyper)graph G in ZFC?

More on smooth graphs

G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that **$G[W \setminus W']$ is isomorphic to G**.

Problem [L. Soukup 1997]

Is there a **nontrivial ω-smooth or ω_1-smooth graph G?**

It is **independent** whether there is a nontrivial **$< \omega$-smooth** graph:

- **No**, if $\omega_1 < \mathfrak{s}$ or $2^\omega < 2^{\omega_1}$ or **in $V^{\mathbb{C}_{\omega_2}}$**.
- **Yes**, such graphs **can be forced**: add a generic graph + add isomorphisms.

More on smooth graphs

G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that **$G[W \setminus W']$ is isomorphic to G**.

Problem [L. Soukup 1997]

Is there a **nontrivial ω-smooth or ω_1-smooth graph G?**

It is **independent** whether there is a nontrivial **$< \omega$-smooth** graph:

- **No**, if $\omega_1 < \mathfrak{s}$ or $2^\omega < 2^{\omega_1}$ or **in $V^{\mathbb{C}_{\omega_2}}$**.
- **Yes**, such graphs **can be forced**: add a generic graph + add isomorphisms.

More on smooth graphs

G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that **$G[W \setminus W']$ is isomorphic to G**.

Problem [L. Soukup 1997]

Is there a **nontrivial ω-smooth or ω_1-smooth graph G?**

It is **independent** whether there is a nontrivial **$< \omega$-smooth** graph:

- **No**, if $\omega_1 < \mathfrak{s}$ or $2^\omega < 2^{\omega_1}$ or **in $V^{\mathbb{C}_{\omega_2}}$**.
- **Yes**, such graphs **can be forced**: add a generic graph + add isomorphisms.

More on smooth graphs

G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that **$G[W \setminus W']$ is isomorphic to G**.

Problem [L. Soukup 1997]

Is there a **nontrivial ω-smooth or ω_1-smooth graph G?**

It is **independent** whether there is a nontrivial **$< \omega$-smooth** graph:

- **No**, if $\omega_1 < \mathfrak{s}$ or $2^\omega < 2^{\omega_1}$ or **in $V^{\mathbb{C}_{\omega_2}}$**.
- **Yes**, such graphs **can be forced**: add a generic graph + add isomorphisms.

More on smooth graphs

G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that **$G[W \setminus W']$ is isomorphic to G**.

Problem [L. Soukup 1997]

Is there a **nontrivial ω-smooth or ω_1-smooth graph G?**

It is **independent** whether there is a nontrivial **$< \omega$-smooth** graph:

- **No**, if $\omega_1 < \mathfrak{s}$ or $2^\omega < 2^{\omega_1}$ or **in** $V^{\mathbb{C}_{\omega_2}}$.
- **Yes**, such graphs **can be forced**: add a generic graph + add isomorphisms.

More on smooth graphs

G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that **$G[W \setminus W']$ is isomorphic to G**.

Problem [L. Soukup 1997]

Is there a **nontrivial ω-smooth or ω_1-smooth graph G?**

It is **independent** whether there is a nontrivial **$< \omega$-smooth** graph:

- **No**, if $\omega_1 < \mathfrak{s}$ or $2^\omega < 2^{\omega_1}$ or **in** $V^{\mathbb{C}_{\omega_2}}$.
- **Yes**, such graphs **can be forced**: add a generic graph + add isomorphisms.

Anti-Ramsey results for graphs

What do you need for Ramsey's theorem to fail?

Todorcevic proved that $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ i.e. we can colour the edges of K_{ω_1} such that **every colour appears on every uncountable induced subgraph**.

Conjecture [Erdős, Galvin, Hajnal 1975]

Suppose that $G = (V, E)$ is an ω_1-chromatic graph. Then **there is a colouring** $c : E \to \omega_1$ such that for every $V = \bigcup_{i \in \omega} V_i$ there is $i < \omega$ such that F **assumes all values on** $G[V_i]$.

Anti-Ramsey results for graphs

What do you need for Ramsey's theorem to fail?

Todorcevic proved that $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ i.e. we can colour the edges of K_{ω_1} such that **every colour appears on every uncountable induced subgraph**.

Conjecture [Erdős, Galvin, Hajnal 1975]

Suppose that $G = (V, E)$ is an ω_1-chromatic graph. Then **there is a colouring** $c : E \to \omega_1$ such that for every $V = \bigcup_{i\in\omega} V_i$ there is $i < \omega$ such that F **assumes all values on** $G[V_i]$.

Anti-Ramsey results for graphs

What do you need for Ramsey's theorem to fail?

Todorcevic proved that $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ i.e. we can colour the edges of K_{ω_1} such that **every colour appears on every uncountable induced subgraph**.

Conjecture [Erdős, Galvin, Hajnal 1975]

Suppose that $G = (V, E)$ is an ω_1-chromatic graph. Then **there is a colouring** $c : E \to \omega_1$ such that for every $V = \bigcup_{i \in \omega} V_i$ there is $i < \omega$ such that F **assumes all values on** $G[V_i]$.

Anti-Ramsey results for graphs

What do we know?

[Hajnal, Komjáth 2003] Yes if a single Cohen real is added.

- $\diamondsuit^+$ implies the conjecture for graphs of size ω_1 in a strong sense: there is a colouring such that every colour appears on every subgraph with chromatic number ω_1.
- Consistently, there is a graph such that every colouring is constant on an uncountably chromatic subgraph.

What do we know?

[Hajnal, Komjáth 2003] **Yes** if a single Cohen real is added.

- $\diamondsuit^+$ implies the conjecture for graphs of size ω_1 in a strong sense: there is a colouring such that **every colour** appears on **every subgraph** with chromatic number ω_1.
- **Consistently**, there is a graph such that **every colouring is constant** on an uncountably chromatic subgraph.

What do we know?

[Hajnal, Komjáth 2003] **Yes** if a single Cohen real is added.

- $\diamondsuit^+$ implies the conjecture for graphs of size ω_1 in a strong sense: there is a colouring such that **every colour** appears on **every subgraph** with chromatic number ω_1.
- **Consistently**, there is a graph such that **every colouring is constant** on an uncountably chromatic subgraph.

Anti-Ramsey results for graphs

What do we know?

[Hajnal, Komjáth 2003] **Yes** if a single Cohen real is added.

- $\diamondsuit^+$ implies the conjecture for graphs of size ω_1 in a strong sense: there is a colouring such that **every colour** appears on **every subgraph** with chromatic number ω_1.
- **Consistently**, there is a graph such that **every colouring is constant** on an uncountably chromatic subgraph.

Anti-Ramsey results for graphs - how to proceed?

Prove the (strong) conjecture for certain classes of graphs!

Let $(T)^2 = \{\{s, t\} : s \leq t \in T\}$ for a **tree** T.

Conjecture

For every **non special tree** T there is a colouring $c : (T)^2 \to \omega_1$ such that $c''(S)^2 = \omega_1$ for every non special subtree S.

This is true for non special trees T if:

- T is **Souslin**, or
- if **MA** holds, $|T| = 2^\omega$ and T has no uncountable chains.

Anti-Ramsey results for graphs - how to proceed?

Prove the (strong) conjecture for certain classes of graphs!

Let $(T)^2 = \{\{s,t\} : s \leq t \in T\}$ for a **tree** T.

Conjecture

For every **non special tree** T there is a colouring $c : (T)^2 \to \omega_1$ such that $c''(S)^2 = \omega_1$ for every non special subtree S.

This is true for non special trees T if:

- T is **Souslin**, or
- if **MA** holds, $|T| = 2^\omega$ and T has no uncountable chains.

Anti-Ramsey results for graphs - how to proceed?

Prove the (strong) conjecture for certain classes of graphs!

Let $(T)^2 = \{\{s, t\} : s \leq t \in T\}$ for a **tree** T.

Conjecture

For every **non special tree** T there is a colouring $c : (T)^2 \to \omega_1$ such that $c''(S)^2 = \omega_1$ for every non special subtree S.

This is true for non special trees T if:

- T is **Souslin**, or
- if **MA** holds, $|T| = 2^\omega$ and T has no uncountable chains.

Anti-Ramsey results for graphs - how to proceed?

Prove the (strong) conjecture for certain classes of graphs!

Let $(T)^2 = \{\{s,t\} : s \leq t \in T\}$ for a **tree** T.

Conjecture

For every **non special tree** T there is a colouring $c : (T)^2 \to \omega_1$ such that $c''(S)^2 = \omega_1$ for every non special subtree S.

This is true for non special trees T if:

- T is **Souslin**, or
- if MA holds, $|T| = 2^\omega$ and T has no uncountable chains.

Anti-Ramsey results for graphs - how to proceed?

Prove the (strong) conjecture for certain classes of graphs!

Let $(T)^2 = \{\{s,t\} : s \leq t \in T\}$ for a **tree** T.

Conjecture

For every **non special tree** T there is a colouring $c : (T)^2 \to \omega_1$ such that $c''(S)^2 = \omega_1$ for every non special subtree S.

This is true for non special trees T if:

- T is **Souslin**, or
- if MA holds, $|T| = 2^\omega$ and T has no uncountable chains.

Topics I should have mentioned...

- **incompactness** of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^{\kappa} Chr(H) = \omega_1)\} = ???$
- theory of **definable graphs** (closed/open graphs);
- **Borel**/analytic **chromatic number** (S. Geschke, Kechris, S. Todorcevic, ...);
- Find **gaps**, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is **no** $G \to H$ **homomorphism** (N. Sauer, S. Shelah...);
- problems on **countable graphs** (e.g. R. Diestel and Hamburg school).

Topics I should have mentioned...

- **incompactness** of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^\kappa Chr(H) = \omega_1)\} =???$
- theory of **definable graphs** (closed/open graphs);
- **Borel**/analytic **chromatic number** (S. Geschke, Kechris, S. Todorcevic, ...);
- Find **gaps**, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is **no** $G \to H$ **homomorphism** (N. Sauer, S. Shelah...);
- problems on **countable graphs** (e.g. R. Diestel and Hamburg school).

Topics I should have mentioned...

- **incompactness** of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^\kappa Chr(H) = \omega_1)\} =???$
- theory of **definable graphs** (closed/open graphs);
- **Borel**/analytic **chromatic number** (S. Geschke, Kechris, S. Todorcevic, ...);
- Find **gaps**, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is **no** $G \to H$ **homomorphism** (N. Sauer, S. Shelah...);
- problems on **countable graphs** (e.g. R. Diestel and Hamburg school).

Topics I should have mentioned...

- incompactness of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^\kappa Chr(H) = \omega_1)\} =???$
- theory of definable graphs (closed/open graphs);
- Borel/analytic chromatic number (S. Geschke, Kechris, S. Todorcevic, ...);
- Find gaps, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is no $G \to H$ homomorphism (N. Sauer, S. Shelah...);
- problems on countable graphs (e.g. R. Diestel and Hamburg school).

Topics I should have mentioned...

- **incompactness** of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^\kappa Chr(H) = \omega_1)\} =???$
- theory of **definable graphs** (closed/open graphs);
- **Borel**/analytic **chromatic number** (S. Geschke, Kechris, S. Todorcevic, ...);
- Find **gaps**, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is **no** $G \to H$ **homomorphism** (N. Sauer, S. Shelah...);
- problems on **countable graphs** (e.g. R. Diestel and Hamburg school).

Topics I should have mentioned...

- **incompactness** of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^\kappa Chr(H) = \omega_1)\} =???$
- theory of **definable graphs** (closed/open graphs);
- **Borel**/analytic **chromatic number** (S. Geschke, Kechris, S. Todorcevic, ...);
- Find **gaps**, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is **no** $G \to H$ **homomorphism** (N. Sauer, S. Shelah...);
- problems on **countable graphs** (e.g. R. Diestel and Hamburg school).

Topics I should have mentioned...

- **incompactness** of chromatic number (see recently A. Rinot),
  - $\min\{\kappa : \forall G(Chr(G) = \omega_1 \Rightarrow \exists H \in [G]^\kappa Chr(H) = \omega_1)\} =???$
- theory of **definable graphs** (closed/open graphs);
- **Borel**/analytic **chromatic number** (S. Geschke, Kechris, S. Todorcevic, ...);
- Find **gaps**, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is **no** $G \to H$ **homomorphism** (N. Sauer, S. Shelah...);
- problems on **countable graphs** (e.g. R. Diestel and Hamburg school).

Thank you for your attention.

Any questions?