OPEN PROBLEMS AROUND UNCOUNTABLE GRAPHS

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Abstract. We collect some famous and some less well know open problems from the theory of uncountable graphs focusing mainly on chromatic number. This short note serves as a reference list for a talk given at the University of East Anglia, *Independence Results in Mathematics and Challenges in Iterated Forcing* workshop in November, 2015.

We highlight that there are several very well written recent surveys on problems in infinite combinatorics and infinite graphs: Komjáth [18, 20], Todorcevic [32]. Although we mainly focus on uncountable graphs, there is significant work done on countably infinite graphs: see the survey of Halin [11] and the projects of the Hamburg school lead by R. Diestel [3].

We are grateful for the help of P. Komjáth, L. Soukup, S. Todorcevic and many others in preparing these notes.

Basic definitions. A graph G is an ordered pair (V, E) such that $E \subseteq [V]^2$. We let G^C denote **the complement of** G i.e. $(V, [V]^2 \setminus E)$. We let $N_G(v) = \{u \in V : \{u, v\} \in E\}$ for $v \in V$.

The **chromatic number** of a graph G is the minimal cardinal κ such that G admits a **good colouring** i.e. there is $f : V \to \kappa$ such that $f(x) \neq f(y)$ whenever $\{x, y\} \in E$.

The **product** $G_0 \times G_1$ of $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ has vertices $V_0 \times V_1$ and edges $\{(x, y), (x', y')\}$ where $\{x, x'\} \in E_0$ and $\{y, y'\} \in E_1$.

1. Connected subgraphs

A graph G is **infinitely connected** iff the removal of finitely many vertices leaves G connected.

Conjecture 1.1 (Erdős, Hajnal)**.** *There is a graph G with chromatic number ω_1 without non empty infinitely connected subgraphs.*

There is a graph with chromatic number ω_1 without *uncountable* infinitely connected subgraphs [29] (but the known examples contain plenty of countably infinite complete subgraphs). A version of this problem first appeared in [6] and later in several surveys and problem sets [7, 18, 20]. Related results are proved in [15, 16, 19].

2. Triangle free subgraphs

Conjecture 2.1 (Erdős, Hajnal)**.** *There is a graph G with chromatic number ω_1 such that every subgraph with uncountable chromatic number contains a triangle.*

Consistently yes (even K_4 can be ommitted in G, while CH may or may not hold) [21]. It would be interesting to find more constructive examples, e.g. using $\diamondsuit$. The problem is still open in ZFC.

3. On the growth of finite subgraphs

Conjecture 3.1 (Erdős)**.** *Suppose $f:\mathbb{N}\to\mathbb{N}$ is increasing. Then there is a graph G with uncountable chromatic number such that every n chromatic subgraph of G has size at least $f(n)$.*

Consistently yes [22] but still open in ZFC.

4. Common subgraphs

Problem 4.1 (Erdős)**.** *Does every two graph with uncountable chromatic number (chromatic number ω_1) contain a common*
(a) 4-chromatic subgraph?
(b) ω-chromatic subgraph?

Yes, for 3-chromatic subgraphs (there is a common odd cycle). Also, no for ω_1-chromatic subgraphs: take G_0, G_1 with $Chr(G_0\times G_1)\le\omega$. On a related note, the following could be interesting:

Problem 4.2 (L. Soukup)**.** *Construct G_0 and G_1 such that $Chr(G_0\times G_1)=\omega_1$ but G_0 and G_1 has no common ω_1-chromatic subgraphs.*

The following might be much easier to answer then Erdős' original question:

Problem 4.3 (D.T. Soukup)**.** *Suppose G is a graph on ω_1 such that both G and its complement has uncountable chromatic number. Is there a common ω_1-chromatic subgraph of G and G complement?*

5. Hajnal-Máté graphs

An **HM-graph** is a graph G with $V(G)=Chr(G)=\omega_1$ such that $N_G(\beta)\cap\alpha$ is finite for all $\alpha<\beta<\omega_1$ [9]. $\diamondsuit$ implies the existence of (triangle free) HM-graphs while MA_{ω_1} negates the existence.

I expect that a recently developed technique of Mildenberg and Shelah [25] (proving that $\clubsuit$ can hold in a model where every Aronszajn tree is special) can be understood by and adapted to solve the following:

Problem 5.1 (D. T. Soukup)**.** *Is it consistent that $\clubsuit$ holds and there are no HM graphs on ω_1?*

Using a Cohen real and ladder systems one can construct Suslin trees [31] or HM-graphs. We ask:

Problem 5.2 (D. T. Soukup)**.** *Does adding a single Cohen real imply the existence of a triangle free HM-graph on ω_1?*

See more about HM-graphs in Komjáth's papers [13, 14, 17]. Also, on a generalization see Abraham and Yin [1].

6. Simultaneous chromatic number

Conjecture 6.1 (Erdős, Hajnal)**.** *Suppose that (V,E) is an ω_1-chromatic graph. Then there is a colouring $c:E\to\omega_1$ such that for every $V=\bigcup_{i\in\omega}V_i$ there is $i<\omega$ such that F assumes all values on V_i.*

Consistently yes [8] e.g. adding a single Cohen real suffices.

It would be interesting to see if at least certain special classes of graphs satisfy this conjecture e.g. HM-graphs or comparability graphs of posets. We formulate the following related conjecture:

Conjecture 6.2 (D. T. Soukup)**.** *Suppose that T is a non special tree and let $(T)^2 = \{\{s,t\} : s \leq t \in T\}$. Then there is a colouring $c : (T)^2 \to \omega_1$ such that whenever $S \subseteq T$ is a non special subtree of T then $c''(S)^2 = \omega_1$.*

The above conjecture holds for Suslin trees and, under MA, for non special trees of size 2^ω without uncountable chains [28].

7. Smooth graphs

G is **non trivial** iff K_{ω_1} does not embed into G and G^C. E.g. the Sierpinksi-graph. G is **almost smooth** iff the induced subgraph $G[W]$ on W is isomorphic to G whenever $|V \setminus W| < |V|$ [12]. In [12], the authors show that every graph which is isomorphic to all of its uncountable induced subgraphs is either the empty or the complete graph. Hence

Problem 7.1 (Kierstead, Nyikos 1989)**.** *Is there in ZFC a non trivial almost smooth graph on ω_1?*

Consistently yes (CH or forcing) [10].

A graph G on ω_1 is **κ-smooth** iff for every uncountable $W \subseteq \omega_1$ there is a $|W'| \leq \kappa$ such that $G[W \setminus W']$ is isomorphic to G.

Problem 7.2 (L. Soukup 1997)**.** *Is there in ZFC a nontrivial ω-smooth or ω_1-smooth graph G?*

It is independent whether there is a nontrivial $< \omega$-smooth graph [30]. Another related paper is [27].

8. Products

Problem 8.1 (L. Soukup)**.** *Is there a graph G on ω_1 such that $Chr(G) = Chr(G^C) = \omega_1$ but*

$$Chr(G \times G^C) \leq \omega.$$

No consistency results are known. Otherwise, there are (in ZFC) G_0, G_1 on ω_1 such that $Chr(G_0) = Chr(G_1) = \omega_1$ but $Chr(G_0 \times G_1) \leq \omega$.

9. Unfriendly partitions

Problem 9.1 (R. Cowan, W. Emerson)**.** *Is it true that every countable graph (V, E) admits an* ***unfriendly partition*** *i.e. there is $V = V_0 \cup V_1$ such that*

$$|N(x) \cap V_{1-i}| \geq |N(x) \cap V_i|$$

for every $x \in V_i$ and $i < 2$.

Every finite graph and every countable graph with finite degrees or purely infinite degrees has an unfriendly partition. Also, every graph with finitely many vertices of infinite degree has an unfriendly partition [2].

The following are proved in [26]: uncountable graphs may not have unfriendly partitions but the smallest known examples have size $(2^\omega)^{+\omega}$. Interestingly, every graph has an unfriendly 3-partition.

10. More on girth and chromatic number

We write $Chr(G) > H$ iff there is no homomorphism from G to H. The following would significantly extend the well known theorem of Erdős [5] on girth and chromatic number:

Problem 10.1 (L. Soukup[1])**.** *Is it true, that for every non bipartite H without K_ω and every $l \in \omega$ there is a G with $Chr(G) > H$ and girth bigger than l?*

It would be nice learn more information about antichains, minimal elements in certain classes of graphs $\mathcal{G}$ quasi-ordered by $H < G$ iff there is no homomorphism from G to H. See [23, 4] for related results.

11. Other topics of interest

We list a few topics here which are of great interest:

- incompactness of chromatic number (see recently A. Rinot [24]),
- theory of definable graphs (closed/open graphs);
- Borel/analytic chromatic number (S. Geschke, Kechris, S. Todorcevic, ...);
- Find gaps, finite maximal antichains, etc. in $(\mathcal{G}, \prec)$ where $H \prec G$ iff there is $G \to H$ homomorphism (N. Sauer, S. Shelah...);
- problems on countable graphs (e.g. R. Diestel and Hamburg school).

[1]personal communication

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