AROUND D-SPACES

RECENT PROGRESS IN THE THEORY OF COVERING PROPERTIES

Dániel Tamás Soukup

MSc Thesis

Supervisor: Zoltán Szentmiklóssy, assistant professor

Eötvös Loránd University

June 2011, Budapest

Contents

0.1 Introduction 2

1 D-spaces **5**

1.1 Basic results 6

1.2 Linearly D-spaces 7

1.3 aD-spaces 8

1.4 Various open problems 9

2 Is every Lindelöf space a D-space? **10**

2.1 On Lindelöf implies D 11

2.1.1 Menger spaces 11

2.1.2 Subparacompact spaces 13

2.2 On Lindelöf does not imply D 16

2.2.1 Preliminaries 16

2.2.2 The construction 17

2.3 Remarks 19

3 Properties D and aD **20**

3.1 Questions and answers 20

3.2 Notes on MAD families 21

3.3 Fragments of Shelah's club guessing 22

3.4 The general construction 23

3.5 Focusing on property D and aD 25

3.6 Examples of aD, non linearly D-spaces 29

3.7 Consistently on property D and aD 30

3.7.1 Preliminaries 30

3.7.2 The main result 32

0.1 Introduction

D-spaces were introduced approximately 40 years ago as a straightforward generalization of compactness. Since then, D-spaces are in the center of interest for general topologists. However, we still lack understanding of the relationship of classical covering properties to D-spaces. The purpose of this thesis is to gather some of the most recent results from the theory of D-spaces and related covering properties.

In Chapter 1 we introduce the notion of D-spaces and tend to make the reader familiar with the classical results. A further aim of this chapter is to define properties closely related to property D; that is, we introduce linearly D-spaces, aD-spaces, and we state some theorems. Finally, Chapter 1 ends with a short list of open problems concerning D-spaces.

In Chapter 2 our aim is to present the recent progress on the long standing problem whether Lindelöf implies D. First, we present the main result of [4] which is one of the breakthroughs in the topic:

Theorem 2.1.4 . *Every Menger space is a D-space.*

This theorem is a nice application of topological games to the theory of D-spaces. As a corollary, we show the following:

Corollary 2.1.6 . *Martin's Axiom implies that every Lindelöf space of size less than 2^ω is D.*

Following this line of research, the previous corollary was made stronger by D. Repovš, L. Zdomskyy in [23] and W. Shi, H. Zhang in [26] independently.

Theorem 2.1.18 . *Consistently every subparacompact space of size ω_1 is a D-space.*

We outline the methods involved. Opposing these results, we summarize Szeptycki's [30] stating that

Theorem 2.2.1 . $\diamondsuit$ *implies the existence of a T_1-Lindelöf non D-space.*

Finally, Chapter 3 gathers the author's results in the topic. aD-spaces are a well-known generalization of D-spaces; until recently, it was not known whether there is an aD, non D-space. Our main result is Theorem 3.1.1, stating that

Theorem 3.1.1 ([28, Theorem 1.1]). *There exists a 0-dimensional, Hausdorff, scattered aD-space which is not linearly D.*

This answers a question of Arhangel'skii [3] among others; a complete introduction to the problem is in Section 3.1. The proof of the theorem requires advanced set-theoretical methods such as Shelah's club-guessing theory.

Also, we prove the following independence result in Section 3.7.

Theorem 3.1.2 ([27, Theorem 5.2]). *The existence of a locally countable, locally compact space X of size ω_1 which is aD and non linearly D is independent of ZFC.*

The thesis contains certain notions and notations without explicit definitions; let us refer to Engelking's [14] and Kunen's [21] for topological and set-theoretical background, respectively.

I would like to thank the help of the Set Theory and Topology research group at the Rényi Institute in preparing [27] and [28]. Also, I would like to thank Assaf Rinot for his ideas and advices to look deeper into the theory of club guessing in ZFC.

I dedicate this thesis to my father.
Without his support I would not be here.

Chapter 1

D-spaces

An *open neighborhood assignment* (ONA, in short) on a topological space (X, τ) is a map $U : X \to \tau$ such that $x \in U(x)$ for every $x \in X$.

Definition 1.0.1. *X is said to be a D-*space *iff for every neighborhood assignment U on X, there is a closed discrete $D \subseteq X$ such that $X = \bigcup_{d \in D} U(d) = \bigcup U[D]$; such a set D is called a* kernel for U.

The notion of a *D-space* was probably first introduced by van Douwen and E. Michael in the mid-1970's; Michael sent van Douwen a letter with a proof that semistratifiable spaces are D, and van Douwen replied with an alternate proof, in a letter dated June 6, 1975 [15]. Since than, many work had been done it the topic.

Property D can be thought of as a covering property; every compact space, moreover σ-compact space, is a D-space and countably compact D-spaces are compact. Let us quote Gary Gruenhage's words from the inspiring survey [15]:

> *Part of the fascination with D-spaces is that, aside from these easy facts, very little else is known about the relationship between the D property and many of the standard covering properties. For example, it is not known if a very strong covering property such as hereditarily Lindelöf implies D, and yet for all we know it could be that a very weak covering property such as submetacompact or submetalindelöf implies D!*

Indeed, it is not known whether any of the following properties, even if one adds "hereditarily", imply D: Lindelöf, paracompact, ultraparacompact, strongly paracompact, metacompact, metalindelöf, subparacompact, submetacompact, submetalindelöf, paralindelöf, screenable, σ-metacompact.

In this chapter, our aim is to summarize the basic results about D-spaces and corresponding generalizations.

1.1 Basic results

The next proposition is part of the folklore.

Proposition 1.1.1. *Every σ-compact space is a D-space and countably compact D-spaces are compact.*

This implies that $\mathbb{R}$ with the usual Euclidean topology is D and that ω_1 with the order topology is not a D-space.

Let us say a few words about unions of D-spaces.

Proposition 1.1.2 ([7, Proposition 7.]). *If the space X is the countable union of closed D-subspace then X is a D-space.*

Interestingly, it is not known whether the union of two D-spaces is a D-space again. We remark that there is a σ-discrete space which is non D; see the van Douwen-Wicke-space Γ in [12]. Guo and Junnila generalized Proposition 1.1.2.

Theorem 1.1.3 ([17]). *Suppose that $X = \bigcup_{\alpha<\lambda} X_\alpha$ such that X_α is D and $\bigcup_{\alpha'<\alpha} X_{\alpha'}$ is closed for every $\alpha < \lambda$. Then X is a D-space.*

In many cases, spaces with some additional structure are D-spaces; this is the case with generalized metric spaces or spaces with nice bases.

Theorem 1.1.4. *The following are D-spaces:*

(1) semistratifiable spaces [7],

(2) subspaces of symmetrizable spaces [8],

(3) spaces having a point-countable base [2].

The first two properties are both well-known weakenings of metrizability; for definitions see [16]. The proof of "metrizable space are D" works in the first two cases with minor refinements.

Monotone normal spaces generalize both ordered and metric spaces. We have the following classic result from [7].

Theorem 1.1.5. *Every monotone normal D-space is paracompact.*

At the end of this chapter, in Section 1.4, we cite various open problems concerning D-spaces.

1.2 Linearly D-spaces

A straightforward generalization of property D is due to Guo and Junnila [18]. For a space X a cover $\mathcal{U}$ is *monotone* iff it is linearly ordered by inclusion.

Definition 1.2.1. *A space (X, τ) is said to be* linearly D *iff for any ONA $U : X \to \tau$ for which $\{U(x) : x \in X\}$ is monotone, one can find a closed discrete set $D \subseteq X$ such that $X = \bigcup U[D]$.*

The connection between classical covering properties and linear D-property is determined by the next theorem.

Theorem 1.2.2 ([18, Proposition 2.11])**.** *Every submetalindelöf T_1 space is linearly D.*

Submetalindelöfness is a significant weakening of both Lindelöfness and paracompactness.

Definition 1.2.3. *A space X is* submetalindelöf *iff for every open cover $\mathcal{U}$ of X there are $\{\mathcal{U}_n : n \in \omega\}$ open refinements of $\mathcal{U}$ covering X such that for every $x \in X$ there is $n \in \omega$ such that $|\{U \in \mathcal{U}_n : x \in U\}| \leq \omega$.*

We will later use the following characterization of linear D property. A set $D \subseteq X$ is said to be $\mathcal{U}$*-big* for a cover $\mathcal{U}$ iff there is no $U \in \mathcal{U}$ such that $D \subseteq U$.

Theorem 1.2.4 ([18, Theorem 2.2])**.** *The following are equivalent for a T_1-space X:*

1. *X is linearly D,*

2. *for every non-trivial monotone open cover $\mathcal{U}$ of X, there exists a closed discrete $\mathcal{U}$-big set in X.*

We remark that there are linearly D, non D-spaces; for example, any linearly Lindelöf, non Lindelöf space is such. This can be easily seen from the following:

Proposition 1.2.5 ([18]). *A space X is linearly Lindelöf iff X is linearly D and there are no uncountable closed discrete sets in X.*

For various examples of linearly Lindelöf, non Lindelöf spaces see [22].

1.3 aD-spaces

Investigating the properties of D-spaces and the connections between other covering properties led to the definition of *aD-spaces*, defined by Arhangel'skii and Buzyakova in [2].

Definition 1.3.1. *A space (X, τ) is said to be aD iff for each closed $F \subseteq X$ and for each open cover $\mathcal{U}$ of X there is a closed discrete $D \subseteq F$ and $U : D \to \mathcal{U}$ with $x \in U(x)$ for all $x \in D$ such that $F \subseteq \cup U[D]$.*

It is clear that D-spaces are aD. As it turned out, property aD is much more docile than property D.

Theorem 1.3.2 ([3, Theorem 1.15]). *Every submetalindelöf T_1 space is aD.*

Thus, until the recent results of [28], there was the hope to prove $aD \Rightarrow D$ and settle the problems listed in the introduction.

Proving that a space is aD, the notion of an *irreducible space* will play a key role.

Definition 1.3.3. *A space X is* irreducible *iff every open cover $\mathcal{U}$ has a* minimal open refinement *$\mathcal{U}_0$; meaning that no proper subfamily of $\mathcal{U}_0$ covers X.*

In [3] Arhangel'skii showed the following equivalence.

Theorem 1.3.4 ([3, Theorem 1.8]). *A T_1-space X is an aD-space if and only if every closed subspace of X is irreducible.*

In Chapter 3, we deal with existence of aD, non D-spaces.

1.4 Various open problems

First, let us state some open problems asking if certain spaces are D. Of course, the main interest is in the following.

Problem 1.4.1. *Is every (hereditarily) Lindelöf space a D-space?*

We will deal with this question in Chapter 2. Let us remark again, that it is not known whether any of the following strong covering properties imply D: Lindelöf, paracompact, ultraparacompact, strongly paracompact; similarly, it is unknown whether there is a non D-space with any of the following weak covering properties: metacompact, metalindelöf, subparacompact, submetacompact, submetalindelöf, paralindelöf, screenable, σ-metacompact.

Another main question in the area is the following from Arhangel'skii [3].

Problem 1.4.2. *Is the union of two D-spaces a D-space again? Is it aD?*

Let us cite this classic problem of Borges and Wehrly connected to Theorem 1.1.5.

Problem 1.4.3 ([7]). *Is every paracompact, monotonically normal space a D-space?*

The following is from [9].

Problem 1.4.4. *Suppose that X is a continuous image of a Lindelöf D-space. Is X a D-space?*

Now, we turn to a possibly easier open problems. Suppose that S is the Sorgenfrey-line. Van Douwen and Pfeffer in [11] proved that S and its finite powers are D-spaces; later, de Caux [10] showed that these space are hereditarily D-spaces. However, the following is still open.

Problem 1.4.5. *([10]) Is S^ω a (hereditarily) D-space?*

Suppose that M is a metric space with topology τ. If τ' refines τ such that every point $p \in M$ has a base B in τ' such that $U \setminus \{p\} \in \tau$ for every $U \in B$ then (M, τ') is called a *butterfly space over the metric space* (M, τ). For example, the Sorgenfrey line or its finite powers are butterfly spaces over the appropriate Euclidean metric spaces. The following question is from [15].

Problem 1.4.6. *Is every butterfly space over a separable metric space a D-space?*

The reader can find many more fascinating questions in [13] and in the survey [15].

Chapter 2

Is every Lindelöf space a D-space?

In their article [20] in Open Problems in Topology II, M. Hrušák and J. T. Moore listed twenty open problems from set theoretic topology which should be at the center of research interest. Problem 1.4.1, whether every Lindelöf space is D, is number fourteen on their list.

Until very recently, there were no progress in solving the above problem, nor a preferred conjecture; it seemed to be plausible to exist a Lindelöf, non D-space and also that every Lindelöf space is D. The aim of this chapter is to present some of the main recent results concerning Lindelöf and D-spaces.

One of the first and most prominent results is L. Aurichi's following theorem.

Theorem 2.1.4 ([4]). *Every Menger space is a D-space.*

From this, we deduce the following:

Corollary 2.1.6 . *MA implies that every Lindelöf space of size less than* 2^ω *is a D-space.*

The latter corollary and methods of L. Aurichi were improved by D. Repovš, L. Zdomskyy in [23] and W. Shi, H. Zhang in [26] independently.

Theorem 2.1.18 ([23, Corollary 2.6] and [26] independently). *It is consistent that every subparacompact space of size* $\aleph_1$ *is a D-space.*

On the other hand, the following was proved by P. Szeptycki.

Theorem 2.2.1 ([30]). *It is consistent that there exists a* T_1*-Lindelöf non D-space of size* $\aleph_1$*.*

The above results might indicate, that Problem 1.4.1 is not decidable in ZFC; however, any such theorem is yet to come. The rest of this chapter will summarize the above listed three results.

2.1 On Lindelöf implies D

In this section, our aim is to gather results stating that certain covering properties imply property D.

2.1.1 Menger spaces

Definition 2.1.1. *A space X is said to be* Menger *iff for every sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ of X there are finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ for every $n \in \omega$ such that $\cup\{\mathcal{V}_n : n \in \omega\}$ covers X.*

Note that every σ-compact space is Menger. On the other hand, every Menger space is Lindelöf; indeed, for any open cover $\mathcal{U}$ apply the definition of being Menger for the constant sequence $\mathcal{U}_n = \mathcal{U}$.

The Menger property has an interesting and non-trivial characterization by topological games.

Definition 2.1.2. *The* Menger-game *on a space X is the following game of length ω, played by two players, Player I and II. In each round $n \in \omega$, Player I chooses an open cover $\mathcal{U}_n$ of X closed under finite unions and Player II responds by choosing $U_n \in \mathcal{U}_n$. Player II wins iff $X = \bigcup_{n\in\omega} U_n$.*

The following was proved by W. Hurewicz [19].

Theorem 2.1.3. *A space X is Menger iff Player I does not have a winning strategy in the Menger-game on X.*

The next theorem was proved by L. Aurichi.

Theorem 2.1.4 ([4])**.** *Every Menger space is a D-space.*

Proof. Let N be any ONA and let us define a strategy for Player I in the Menger-game on X. First let Player I play $\{\cup N[F] : F \in [X]^{<\omega}\}$. If Player II responds by $\cup N[F_0]$ for some $F_0 \in [X]^{<\omega}$ then let Player I play

$$\{\cup N[F_0 \cup F] : F \in [X]^{<\omega}, F \cap (\cup N[F_0]) = \emptyset\}.$$

Then Player II responds by some $F_1 \in [X]^{<\omega}$. In general, suppose that Player II responded by $F_0, ..., F_{n-1} \in [X]^{<\omega}$ till step n; more precisely, the responds are coded by these finite sets. Let $F_{<n} = \cup\{F_i : i < n\}$ and let Player I play

$$\{\cup N[F_{<n} \cup F] : F \in [X]^{<\omega}, F \cap (\cup N[F_{<n}]) = \emptyset\}.$$

This defines a strategy for Player I in the Menger-game. Since X is Menger, this strategy is not winning by Theorem 2.1.3. Thus there are finite subsets F_n of X for $n \in \omega$ such that $\{\cup N[F_{<n}] : n \in \omega\}$ covers X. Let $D = \cup\{F_n : n \in \omega\}$, then $X = \cup N[D]$. It is easy to check that D is closed and discrete in X; hence, X is a D-space. $\square$

Before stating the first consistency results concerning D-spaces, we need a small claim. Let us recall the definition of the *dominating number*; a family $\mathcal{D} \subseteq \omega^\omega$ is called *dominating* iff for every $g \in \omega^\omega$ there is $f \in \mathcal{D}$ such that $g(n) < f(n)$ for all but finitely many $n \in \omega$. The dominating number $\mathfrak{d}$ is a cardinal defined as follows.

$$\mathfrak{d} = \min\{\kappa : \text{ there is a dominating family } \mathcal{D} \subseteq \omega^\omega \text{ of size } \kappa\}$$

It is clear that $\omega < \mathfrak{d} \leq 2^\omega$ and consistently $\mathfrak{d}$ can be smaller than the continuum; it is well known that MA, Martin's Axiom, implies that $\aleph_1 < \mathfrak{d} = 2^\omega$.

Claim 2.1.5. *Every Lindelöf space of size less than $\mathfrak{d}$ is Menger; hence D by Theorem 2.1.4.*

Proof. Let us fix a sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ of X; we can suppose that each $\mathcal{U}_n$ is countable by X being Lindelöf. Let $\mathcal{U}_n = \{U_{n,k} : k \in \omega\}$ for $n \in \omega$. For every $x \in X$ define $f_x : \omega \to \omega$ such that $x \in U_{n,f_x(n)}$ for every $n \in \omega$. $|X| < \mathfrak{d}$ implies that there is a function $g \in \omega^\omega$ such that $f_x(n) \leq g(n)$ for infinitely many $n \in \omega$, for every $x \in X$; observe that $x \in \cup\{U_{n,k} : k \leq g(n)\}$ for such $n \in \omega$. Let $\mathcal{V}_n = \{U_{n,k} : k \leq g(n)\} \in [\mathcal{U}_n]^{<\omega}$ for $n \in \omega$. Clearly, $\cup\{\mathcal{V}_n : n \in \omega\}$ covers X. Thus X is Menger. $\square$

Corollary 2.1.6. *MA implies that every Lindelöf space of size less than 2^ω is a D-space.*

Thus there is no ZFC example of a Lindelöf, non D-space of size less than 2^ω. In the next section, we will reach analogues corollaries concerning paracompact spaces.

Let us mention a similar theorem to Claim 2.1.5 without proof. Let $\mathcal{M}$ denote the ideal of meager sets.

Theorem 2.1.7 ([4]). *If a Lindelöf space X can be covered by fewer than $cov(\mathcal{M})$ many compact sets then X is Menger, hence D.*

Finally, let us cite another application of Theorem 2.1.4. A space X is called *productively Lindelöf* iff $X \times Y$ is Lindelöf for every Lindelöf space Y.

Theorem 2.1.8 ([31, Theorem 2]). *The Continuum Hypothesis implies that every productively Lindelöf space is Menger, hence D.*

As a corollary, under CH every Lindelöf P-space is a D-space.

2.1.2 Subparacompact spaces

Our aim in this section is to present the methods of [23] and the proof of Theorem 2.1.18. If not stated otherwise, the definitions and results are from [23].

Let us introduce a weakening of property D.

Definition 2.1.9. *A space X has* property D_κ or X is a D_κ-space *for some cardinal κ iff for every ONA N on X there are closed discrete subsets $\{D_\xi : \xi < \kappa\}$ of X such that $\cup\{N[D_\xi] : \xi < \kappa\}$ covers X.*

We omit the proof of the following observations.

Claim 2.1.10.

1. *$D = D_n \Rightarrow D_\lambda \Rightarrow D_\kappa$ for every cardinal $\lambda \leq \kappa$ and $0 < n < \omega$.*
2. *Property D_κ is inherited by closed subsets.*
3. *Every Lindelöf space is a D_ω-space.*

Next, we will need a notion very similar to the Menger property.

Definition 2.1.11. *A space X has property E^*_ω iff for every sequence of countable open covers $\{\mathcal{U}_n : n \in \omega\}$ of X there are finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ for every $n \in \omega$ such that $\cup\{\mathcal{V}_n : n \in \omega\}$ covers X.*

Note that property E^*_ω is equivalent to the Menger property in the realm of Lindelöf spaces. Let us define the corresponding *E^*_ω-game*:

Definition 2.1.12. *The* E^*_ω-game *on a space X is the following game of length ω, played by two players, Player I and II. In each round $n \in \omega$, Player I chooses a countable, increasing open cover $\mathcal{U}_n = \{U_{n,k} : k \in \omega\}$ of X and Player II responds by choosing $k_n \in \omega$. Player II wins iff $X = \bigcup_{n\in\omega} U_{n,k_n}$.*

The direct analogue of Theorem 2.1.3 holds in this case.

Proposition 2.1.13. *A space X has property E^*_ω iff Player I has no winning strategy in the E^*_ω-game on X.*

The main theorem of this section is the following.

Theorem 2.1.14 ([23, Theorem 2.1])**.** *Suppose that a space X has properties D_ω and E^*_ω. Then X is a D-space.*

Proof. Let N be any ONA on X and we define a strategy for Player I in the E^*_ω-game. Let $F_0 = X$; F_0 is a D_ω-space so there is an increasing sequence $\{A_{0,k} : k \in \omega\}$ of closed discrete subsets of F_0 such that $\{\cup N[A_{0,k}] : k \in \omega\}$ covers X. Let Player I play $\{\cup N[A_{0,k}] : k \in \omega\}$. Player II responds by choosing A_{0,k_0} for some $k_0 \in \omega$. Let $F_1 = X \setminus \cup N[A_{0,k_0}]$, then there is an increasing sequence $\{A_{1,k} : k \in \omega\}$ of closed discrete subsets of F_1 such that $\{\cup N[A_{1,k}] : k \in \omega\}$ covers F_1. Let Player I play $\{X \setminus F_1 \cup \bigcup N[A_{1,k}] : k \in \omega\}$. Player II responds by choosing A_{1,k_1} for some $k_1 \in \omega$. In general, let $F_n = X \setminus \bigcup_{i<n} \cup N[A_{i,k_i}]$. F_n is closed in X, hence a D_ω-space; thus there is an increasing sequence $\{A_{n,k} : k \in \omega\}$ of closed discrete subsets of F_n such that $\{\cup N[A_{n,k}] : k \in \omega\}$ covers F_n. Let Player I play $\{X \setminus F_n \cup \bigcup N[A_{n,k}] : k \in \omega\}$. Player II responds by choosing A_{n,k_n} for some $k_n \in \omega$.

This strategy is not winning for Player I, hence there are choices $\{k_n : n \in \omega\}$ for Player II such that

$$X = \bigcup_{n\in\omega} \big(X \setminus F_n \cup (\cup N[A_{n,k_n}])\big).$$

Then $X = \bigcup_{n\in\omega} \cup N[A_{n,k_n}]$ since $X \setminus F_0 = \emptyset$ and $X \setminus F_n = \bigcup_{i<n} \cup N[A_{i,k_i}]$ for $n > 0$. Finally, it is easy to see that $\bigcup_{n\in\omega} A_{n,k_n}$ is closed discrete in X. $\square$

Subparacompactness is a significant weakening of paracompactness.

Definition 2.1.15. *A space X is* subparacompact *iff every open cover of X has a σ-locally finite closed refinement.*

Note that not every D_{ω_1} space is a D_ω space; indeed, take ω_1 with the usual order topology. However, we have the following.

Lemma 2.1.16 ([23, Lemma 2.3])**.** *Suppose that X is a subparacompact space which can be covered by ω_1-many of its Lindelöf subspaces. Then X is a D_ω space. In particular, every subparacompact space of size ω_1 is a D_ω-space.*

Proof. Suppose that $X = \bigcup_{\xi<\omega_1} L_\xi$ such that L_ξ is Lindelöf for all $\xi < \omega_1$; without the loss of generality, we can suppose that $L_\xi \subseteq L_\eta$ for all $\xi < \eta < \omega_1$. Let N be any ONA on X. There are countable subsets $\{C_\alpha : \alpha < \omega_1\}$ of X such that

(i) $L_0 \subseteq \cup N[C_0]$,

(ii) $C_\alpha \cap \bigcup_{\xi<\alpha} \cup N[C_\xi] = \emptyset$ for all $\alpha < \omega_1$,

(iii) $L_\alpha \setminus \bigcup_{\xi<\alpha} \cup N[C_\xi] \subseteq \cup N[C_\alpha]$ for all $\alpha < \omega_1$.

Let $C = \bigcup_{\alpha<\omega_1} C_\alpha$; it suffices to prove that C is σ-closed discrete. The subparacompactness of X implies that there is a closed refinement $\mathcal{F} = \cup_{n\in\omega}\mathcal{F}_n$ of the open cover $\{N(x) : x \in C\}$ such that $\mathcal{F}_n$ is locally finite for every $n \in \omega$. Clearly, $F \cap C$ is countable for every $F \in \mathcal{F}$. Let $C \cap F = \{x_{n,F,m} : m \in \omega\}$ for $F \in \mathcal{F}_n$ for some $n \in \omega$ if $C \cap F$ is nonempty. Let $A_{n,m} = \{x_{n,F,m} : F \in \mathcal{F}_n, C \cap F \neq \emptyset\}$; it is easy to see that $A_{n,m}$ is closed discrete and $C = \bigcup_{n,m\in\omega} A_{n,m}$. $\square$

Observe that the proof of Claim 2.1.5 actually gave us the following: every space X of size less than $\mathfrak{d}$ is E^*_ω and every Lindelöf, E^*_ω-space is Menger. Thus, we have the following corollary.

Corollary 2.1.17. *Suppose that X is a subparacompact space of size less than $\mathfrak{d}$ which can be covered by ω_1-many of its Lindelöf subspaces. Then X is a D-space.*

Proof. X is E^*_ω by the previous observation and D_ω by Lemma 2.1.16. Thus X is a D-space by Theorem 2.1.14. $\square$

Thus, we have proved the following either.

Theorem 2.1.18 ([23, Corollary 2.6] and [26] independently)**.** *MA implies that every subparacompact space of size ω_1 is a D-space.*

Although, we do not know the answer to the following.

Problem 2.1.19. *Is it consistent, that $\omega_1 < 2^\omega$ and every (sub)paracompact space of size less than 2^ω is a D-space?*

2.2 On Lindelöf does not imply D

In this section, we outline a construction of Paul Szeptycki and deduce the following:

Theorem 2.2.1 ([30])**.** *It is consistent that there exists a T_1-Lindelöf non D-space of size $\aleph_1$.*

We skip most of the proofs since the methods involved are greatly advanced. If not stated otherwise, results are from [30].

2.2.1 Preliminaries

The following lemma will be used later.

Lemma 2.2.2. *Consider a topology on ω_1 generated by sets $\{U_\gamma : \gamma < \omega_1\}$ as a subbase; sets of the form*

$$U_F \setminus H \text{ where } U_F = \bigcap\{U_\gamma : \gamma \in F\}$$

for $F, H \in [\omega_1]^{<\omega}$ form a base. If for every uncountable family $B \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets there is a countable $B' \subseteq B$ such that

$$|\omega_1 \setminus \bigcup\{U_F : F \in B'\}| \leq \omega$$

then the topology is hereditarily T_1-Lindelöf.

The construction uses a well known set theoretical principal: Jensen's $\diamondsuit$.

Definition 2.2.3. *A* $\diamondsuit$-sequence *is a sequence $\{S_\beta : \beta < \omega_1\}$ of subsets of ω_1 such that for every $S \subseteq \omega_1$ there are stationary many $\beta \in \omega_1$ such that $S \cap \beta = S_\beta$. Let $\diamondsuit$ denote the statement that there exist a $\diamondsuit$-sequence.*

We need the following observation which is part of the folklore.

Claim 2.2.4. *$\diamondsuit$ is equivalent to the following statement: there exists a sequence $\{B_\beta : \beta < \omega_1\}$ such that $B_\beta \subseteq [\omega_1]^{<\omega}$ for all $\beta < \omega_1$ and for every $B \subseteq [\omega_1]^{<\omega}$ there are stationary many $\beta < \omega_1$ such that $B \cap [\beta]^{<\omega} = B_\beta$.*

2.2.2 The construction

From now on, we assume that $\diamondsuit$ holds; that is, we can fix a sequence $\{B_\beta : \beta < \omega_1\}$ provided by Claim 2.2.4. Also, since $\diamondsuit$ implies the Continuum Hypothesis, we can fix an enumeration $\{C_\alpha : \alpha < \omega_1\}$ of countable subsets of ω_1 such that $C_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$.

Our goals are to construct sets $\{U_\gamma : \gamma < \omega_1\}$ such that $\gamma \in U_\gamma$ for every $\gamma < \omega_1$ and consider the topology on ω_1 generated by this family and the cofinite sets; we will apply Lemma 2.2.2 to prove hereditarily T_1-Lindelöfness and the neighborhood assignment mapping γ to U_γ will show that the space is not a D-space. The next theorem will be the key to achieve our goals.

Theorem 2.2.5. *There exist $\{U^\alpha_\gamma\}_{\gamma\leq\alpha}$ for $\alpha < \omega_1$ with the following properties:*

***IH**(1)* $U^\alpha_\gamma \subseteq \alpha + 1$ *and* $U^\alpha_\alpha = \alpha + 1$ *for every* $\gamma \leq \alpha < \omega_1$.

***IH**(2)* $U^\alpha_\gamma = U^{\alpha_0}_\gamma \cap (\alpha + 1)$.

Let τ_α denote the T_1 topology on $\alpha + 1$ generated by the sets

$$U^\alpha_F = \bigcap\{U^\alpha_\gamma : \gamma \in F\}$$

for $F \in [\alpha + 1]^{<\omega}$ and the cofinite sets of $\alpha + 1$.

***IH**(3)* *If C_α is τ_α closed discrete then* $\bigcup\{U^\alpha_\gamma : \gamma \in C_\alpha\} \neq \alpha + 1$.

***IH**(4)* *Let $T_\alpha = \{\beta \leq \alpha : B_\beta$ is a pairwise disjoint family of finite subsets of β and there is a countable elementary submodel $M \prec H(\aleph_2)$ such that*

- $M \cap \omega_1 = \beta$
- $\{B_\gamma\}_{\gamma<\omega_1} \in M$
- *there is an uncountable $B \in M$ such that $M \cap B = B_\beta$, and*
- *there is $\{V_\gamma\}_{\gamma<\omega_1} \in M$ such that $V_\gamma \cap \beta = U^\alpha_\gamma \cap \beta$ for all $\beta < \alpha\}$.*

(a) If $\beta \in T_\alpha$ then B_β is a local π-network at β in τ_α.

(b) If $\beta \in T_\alpha \cap \alpha$ then for every $V \in \tau_\alpha$ such that $\beta \in V$

$$\{U_F^\alpha : F \in B_\beta, F \subseteq V\}$$

is an ω-cover of $(\beta, \alpha]$.

The proof of Theorem 3.7.6 is done by induction on $\alpha < \omega_1$ while **IH**(1)-**IH**(4) are working as inductive hypothesises. We will not present the proof here; it involves the delicate use of elementary submodels in topology which is beyond the scope of this thesis.

Let us prove now Theorem 2.2.1.

Proof of Theorem 2.2.1. Consider sets $\{U_\gamma^\alpha\}_{\gamma \leq \alpha}$ for $\alpha < \omega_1$ provided by Theorem 3.7.6 with properties **IH**(1)-**IH**(4). Let $U_\gamma = \cup\{U_\gamma^\alpha : \gamma \leq \alpha < \omega_1\}$ for $\gamma < \omega_1$. Let τ denote the topology on ω_1 generated by the sets

$$U_F = \bigcap\{U_\gamma : \gamma \in F\}$$

for $F \in [\omega_1]^{<\omega}$ and the cofinite sets of ω_1.

Lemma 2.2.6. *The topology τ on ω_1 is hereditarily T_1-Lindelöf.*

Proof. We apply Lemma 2.2.2; fix some uncountable family $B \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets. There is an $M \prec H(\aleph_2)$ such that $B, \{U_\gamma : \gamma < \omega_1\}, \{B_\gamma : \gamma < \omega_1\} \in M$ and

$$M \cap \omega_1 = \beta \text{ and } B \cap M = B \cap [\beta]^{<\omega} = B_\beta.$$

We claim that $\omega_1 \setminus \bigcup\{U_F : F \in B_\beta\} \subseteq \beta + 1$; indeed fix some $\alpha \in (\beta, \omega_1)$. Then $\beta \in T_\alpha$, ensured by the model M, and hence there is some $F \in B_\beta$ such that $\alpha \in U_F^\alpha \subseteq U_F$ by **IH**(4). □

Now we prove that (ω_1, τ) is not a D-space. Consider the neighborhood assignment $\gamma \mapsto U_\gamma$; we show that $\cup\{U_\gamma : \gamma \in C\} \neq \omega_1$ for every closed discrete $C \subseteq \omega_1$. Since (ω_1, τ) is T_1-Lindelöf, $|C| \leq \omega$ and hence there is $\alpha < \omega_1$ such that $C_\alpha = C$. It suffices to note that C_α is τ_α closed discrete if τ closed discrete; indeed, then $\cup\{U_\gamma : \gamma \in C_\alpha\} \neq \alpha + 1$ by **IH**(3). □

Whether one can modify the above construction such that the sets $\{U_\gamma : \gamma < \omega_1\}$ are clopen is of central interest.

2.3 Remarks

The Szeptycki-construction gives us only a T_1 example, not even a Hausdorff space; although, it is a great step in solving the main problem, we cannot consider the result a complete (consistent) answer to Problem 1.4.1.

The same holds for the Aurichi and Repovš-Zdomskyy theorems (Theorem 2.1.4 and 2.1.18, respectively). They are outstanding contributions to the investigations done it the topic, however they only provide partial answers.

Thus, the question remains open: Is there a Lindelöf or paracompact, non D-space?

Chapter 3

Properties D and aD

In Section 1.3, we introduced property aD and stated a few facts; every D-space is an aD-space, and even spaces with rather weak covering properties are aD, see Theorem 1.3.2. Therefore, it is worth studying whether there is an aD-space which is not a D-space; a negative answer to this question would settle almost all of the questions about the relationship of classical covering properties to property D.

In this Chapter we answer this question, among others, and we show that there are aD, non D-spaces.

3.1 Questions and answers

In [3] Arhangel'skii asked the following:

Problem 4.6. Is there a Tychonoff aD-space which is not a D-space?

Quite similarly, Guo and Junnila in [18] asked the following about a weakening of property D:

Problem 2.12. Is every aD-space linearly D?

In G. Gruenhage's survey on D-spaces [15], another version of this question is stated (besides the original Arhangel'skii), namely:

Question 3.6(2) Is every scattered, aD-space a D-space?

The main results of this Chapter are following answers to the questions above.

Theorem 3.1.1 ([28, Theorem 1.1]). *There exists a 0-dimensional T_2 space X such that X is scattered, aD, and non linearly D.*

Theorem 3.1.2 ([27, Theorem 5.2]). *The existence of a locally countable, locally compact space X of size ω_1 which is aD and non linearly D is independent of ZFC.*

First, we prove Theorem 3.1.1 as follows; in Section 3.2 and 3.3 we gather all the necessary facts about MAD families and club guessing. In Section 3.4 we define spaces $X[\lambda, \mu, \mathcal{M}, \underline{C}]$, where λ and $\mu = cf(\mu)$ are cardinals, $\mathcal{M}$ is a MAD family on μ, and $\underline{C}$ is a guessing sequence. It is shown in Claim 3.4.2 that

(0) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is always T_2, 0-dimensional, and scattered.

Section 3.5 contains two important results:

(1) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is not linearly D if $cf(\lambda) \geq \mu$ (see Corollary 3.5.3),

(2) $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is aD under certain assumptions (see Corollary 3.5.9).

In Section 3.6 we show how to produce such spaces $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ depending on the cardinal arithmetic and using Shelah's club guessing.

Finally, in Section 3.7 we prove Theorem 3.1.2 using the set theoretical hypothesis ($\diamondsuit^*$) and a result of Zoltán Balogh about "locally nice" spaces under $MA_{\aleph_1}$. We remark, that Section 3.7 can be read independently from the previous sections.

3.2 Notes on MAD families

As MAD families will play an essential part in our constructions we observe some easy facts about them. Let μ be any infinite cardinal. We call $\mathcal{M} \subseteq [\mu]^\mu$ an *almost disjoint family* if $|M \cap N| < \mu$ for all distinct $M, N \in \mathcal{M}$. $\mathcal{M}$ is a *maximal almost disjoint family* (in short, a *MAD family*) if for all $A \in [\mu]^\mu$ there is some $M \in \mathcal{M}$ such that $|A \cap M| = \mu$.

We will use the following rather trivial combinatorial fact.

Claim 3.2.1. *Let $\mathcal{M} \subseteq [\mu]^\mu$ be a MAD family and $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$. Suppose that $N \in [\mu]^\mu$ and $|N \setminus \cup \mathcal{M}'| = \mu$ for all $\mathcal{M}' \in [\mathcal{M}]^{<\mu}$. Then $|\Phi| > \mu$ for $\Phi = \{\varphi < \kappa : |N \cap M^\varphi| = \mu\}$.*

Proof. If $|\Phi| < \mu$ then with $\widetilde{N} = N \setminus \bigcup\{M^\varphi : \varphi \in \Phi\} \in [\mu]^\mu$ we can extend the MAD family, which is a contradiction. If $|\Phi| = \mu$ then let $\Phi = \{\varphi_\zeta : \zeta < \mu\}$. By transfinite induction, construct $\widetilde{N} = \{n_\xi : \xi < \mu\}$ such that $n_\xi \in N \setminus (\bigcup\{M^{\varphi_\zeta} : \zeta < \xi\} \cup \{n_\zeta : \zeta < \xi\})$ for $\xi < \mu$. It is straightforward that $\widetilde{N} \notin \mathcal{M}$ and $\mathcal{M} \cup \{\widetilde{N}\}$ is almost disjoint, which is a contradiction. $\square$

From our point of view the sizes of MAD families are important. Clearly there is a MAD family on ω of size 2^ω. The analogue of this does not always hold for ω_1. Baumgartner in [6] proves that it is consistent with ZFC that there is no almost disjoint family on ω_1 of size 2^{ω_1}. However, we have the following fact.

Claim 3.2.2. *If $2^\omega = \omega_1$ then there is a MAD family $\mathcal{M}$ on ω_1 of size 2^{ω_1}.*

In Section 3.6 we use *nonstationary MAD families $\mathcal{M}_{NS} \subseteq [\mu]^\mu$* meaning that $\mathcal{M}_{NS}$ is a MAD family such that every $M \in \mathcal{M}_{NS}$ is nonstationary in μ. Observe, that using Zorn's lemma to almost disjoint families of nonstationary sets of μ we can get nonstationary MAD families.

3.3 Fragments of Shelah's club guessing

The constructions of the upcoming sections will use the following amazing results of Shelah. For a cardinal λ and a regular cardinal μ let S^λ_μ denote the ordinals in λ with cofinality μ. For an $S \subseteq S^\lambda_\mu$ an *S-club sequence* is a sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that $C_\delta \subseteq \delta$ is a club in δ of order type μ.

Theorem 3.3.1 ([24, Claim 2.3])**.** *Let λ be a cardinal such that $cf(\lambda) \geq \mu^{++}$ for some regular μ and let $S \subseteq S^\lambda_\mu$ stationary. Then there is an S-club sequence $\underline{C} = \langle C_\delta : \delta \in S\rangle$ such that for every club $E \subseteq \lambda$ there is $\delta \in S$ (equivalently, stationary many) such that $C_\delta \subseteq E$.*

A detailed proof of Theorem 3.3.1 can be found in [1, Theorem 2.17].

Theorem 3.3.2 ([25, Claim 3.5])**.** *Let λ be a cardinal such that $\lambda = \mu^+$ for some uncountable, regular μ and $S \subseteq S^\lambda_\mu$ stationary. Then there is an S-club* *sequence $\underline{C} = \langle C_\delta : \delta \in S \rangle$ such that $C_\delta = \{\alpha^\delta_\zeta : \zeta < \mu\} \subseteq \delta$ and for every club $E \subseteq \lambda$ there is $\delta \in S$ (equivalently, stationary many) such that:*

$$\{\zeta < \mu : \alpha^\delta_{\zeta+1} \in E\} \text{ is stationary.}$$

For a detailed proof, see [29].

3.4 The general construction

Definition 3.4.1. *Let $\lambda > \mu = cf(\mu)$ be infinite cardinals. Let $\mathcal{M} \subseteq [\mu]^\mu$ be a MAD family, $\mathcal{M} = \{M^\varphi : \varphi < \kappa\}$ and let $\underline{C} = \{C_\alpha : \alpha \in S^\lambda_\mu\}$ denote an S^λ_μ-club sequence. We define a topological space $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$ as follows. The underlying set of our topology will be a subset of the product $\lambda \times \kappa$. Let*

- $X_\alpha = \{\langle \alpha, 0 \rangle\}$ *for* $\alpha \in \lambda \setminus S^\lambda_\mu$,
- $X_\alpha = \{\alpha\} \times \kappa$ *for* $\alpha \in S^\lambda_\mu$,
- $X = \bigcup\{X_\alpha : \alpha < \lambda\}$.

Let $C_\alpha = \{a^\xi_\alpha : \xi < \mu\}$ denote the increasing enumeration for $\alpha \in S^\lambda_\mu$. For each $\alpha \in S^\lambda_\mu$ let

- $I^\xi_\alpha = (a^\xi_\alpha, a^{\xi+1}_\alpha]$ *for* $\xi \in succ(\mu) \cup \{0\}$,
- $I^\xi_\alpha = [a^\xi_\alpha, a^{\xi+1}_\alpha]$ *for* $\xi \in \lim(\mu)$.

Note that $\bigcup\{I^\xi_\alpha : \xi < \mu\} = (a^0_\alpha, \alpha)$ is a disjoint union.

Define the topology on X by neighborhood bases as follows;

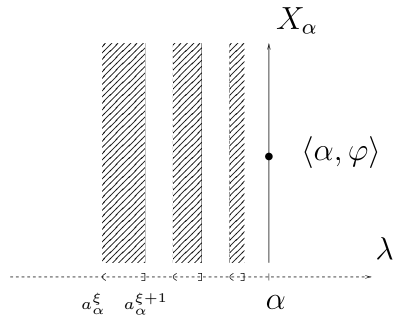
(i) for $\alpha \in S^\lambda_\mu$ and $\varphi < \kappa$ let

$$U(\langle \alpha, \varphi \rangle, \eta) = \{\langle \alpha, \varphi \rangle\} \cup \bigcup\{X_\gamma : \gamma \in \cup\{I^\xi_\alpha : \xi \in M^\varphi \setminus \eta\}\} \text{ for } \eta < \mu$$

and let

$$B(\alpha, \varphi) = \{U(\langle \alpha, \varphi \rangle, \eta) : \eta < \mu\}$$

be a base for the point $\langle \alpha, \varphi \rangle$.



(ii) for $\alpha \in S^\lambda_{<\mu} \cup succ(\lambda) \cup \{0\}$ *let* $\langle \alpha, 0 \rangle$ *be an isolated point,*

(iii) for $\alpha \in S^\lambda_{\mu'}$ *where* $\mu' > \mu$ *let*

$$U(\alpha, \beta) = \bigcup \{X_\gamma : \beta < \gamma \leq \alpha\} \text{ for } \beta < \alpha$$

and let

$$B(\alpha) = \{U(\alpha, \beta) : \beta < \alpha\}$$

be a base for the point $\langle \alpha, 0 \rangle$*.*

It is straightforward to check that these *basic open sets* form neighborhood bases.

⋆

Fix some cardinals $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$, and S^λ_μ-club sequence $\underline{C}$. In the following $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$.

Claim 3.4.2. *The space* $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ *is 0-dimensional,* T_2*, and scattered. Observe that*

(a) X_α *is closed discrete for all* $\alpha < \lambda$*, moreover*

(b) $\bigcup \{X_\alpha : \alpha \in A\}$ *is closed discrete for all* $A \in [\lambda]^{<\mu}$*,*

(c) $X_{\leq \alpha} = \bigcup \{X_\beta : \beta \leq \alpha\}$ *is clopen for all* $\alpha < \lambda$*.*

Proof. First we prove that $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is T_2. Note that

$(*)$ $\bigcup \{X_\gamma : \beta < \gamma \leq \alpha\}$ is clopen for all $\beta < \alpha < \lambda$.

Thus $\langle \alpha, \varphi \rangle, \langle \alpha', \varphi' \rangle \in X$ can be separated trivially if $\alpha \neq \alpha'$. Suppose that $\alpha = \alpha' \in S^\lambda_\mu$ and $\varphi \neq \varphi' < \kappa$. There is $\eta < \mu$ such that $(M^\varphi \cap M^{\varphi'}) \setminus \eta = \emptyset$ since $|M^\varphi \cap M^{\varphi'}| < \mu$. Thus $U(\langle \alpha, \varphi \rangle, \eta) \cap U(\langle \alpha, \varphi' \rangle, \eta) = \emptyset$.

Next we show that $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is 0-dimensional. By $(*)$ it is enough to prove that $U(\langle \alpha, \varphi \rangle, \eta)$ is closed for all $\alpha \in S^\lambda_\mu$, $\varphi < \kappa$ and $\eta < \mu$. Suppose $x = \langle \alpha', \varphi' \rangle \in X \setminus U(\langle \alpha, \varphi \rangle, \eta)$, we want to separate x from $U(\langle \alpha, \varphi \rangle, \eta)$ by an open set. Let $\alpha = \alpha'$. There is $\eta' < \mu$ such that $(M^\varphi \cap M^{\varphi'}) \setminus \eta' = \emptyset$, thus $U(\langle \alpha, \varphi \rangle, \eta) \cap U(\langle \alpha, \varphi' \rangle, \eta') = \emptyset$. Let $\alpha \neq \alpha'$. If $\alpha' \in S^\lambda_{<\mu} \cup \mathrm{succ}(\lambda) \cup \{0\}$ then x is isolated, thus we are done. Suppose $\alpha' \in S^\lambda_{\mu'}$ where $\mu' \geq \mu$. Then $\beta = \sup(C_\alpha \setminus \alpha') < \alpha'$ thus the clopen set $\bigcup \{X_\gamma : \beta < \gamma \leq \alpha'\}$, containing $\langle \alpha', \varphi' \rangle$, is disjoint from $U(\langle \alpha, \varphi \rangle, \eta)$.

$X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is scattered since $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is right separated by the lexicographical ordering on $\lambda \times \kappa$.

(a) and (c) are trivial, we prove (b). Suppose $x = \langle \alpha', \varphi' \rangle \in X$, we prove that there is a neighborhood U of x such that $|U \cap \bigcup \{X_\alpha : \alpha \in A\}| \leq 1$. If $\alpha' \in S^\lambda_{<\mu} \cup \mathrm{succ}(\lambda) \cup \{0\}$ then x is isolated, thus we are done. Suppose $\alpha \in S^\lambda_{\mu'}$ where $\mu' \geq \mu$. Then $\beta = \sup(A \setminus \alpha') < \alpha'$ thus the open set $U = \{x\} \cup \bigcup \{X_\gamma : \beta < \gamma < \alpha\}$ will do the job. $\square$

3.5 Focusing on property D and aD

Again fix some cardinals $\lambda > \mu = cf(\mu)$, a MAD family $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$, and S^λ_μ-club sequence $\underline{C}$. Our next aim is to investigate the spaces $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$ concerning property D and aD.

Definition 3.5.1. *Let $\pi(F) = \{\alpha < \lambda : F \cap X_\alpha \neq \emptyset\}$ for $F \subseteq X$. F is said to be* (un)bounded *if $\pi(F)$ is (un)bounded in λ.*

Let F' denote the set of accumulation points of a subset F of X.

Claim 3.5.2. *If $F \subseteq X$ and $\pi(F)$ accumulates to $\alpha \in S^\lambda_\eta$ such that $\mu \leq \eta < \lambda$ then $F' \cap X_\alpha \neq \emptyset$.*

Proof. If $\eta > \mu$ then $X_\alpha = \{\langle \alpha, 0 \rangle\}$ and each neighborhood $U(\alpha, \beta)$ of $\langle \alpha, 0 \rangle$ intersects F. Thus $F' \cap X_\alpha \neq \emptyset$. Let us suppose that $\pi(F)$ accumulates to $\alpha \in S^\lambda_\mu$. Since $\bigcup \{I^\xi_\alpha : \xi < \mu\} = (a^0_\alpha, \alpha)$, the set $N = \{\xi < \mu : I^\xi_\alpha \cap \pi(F) \neq \emptyset\}$ has cardinality μ. Thus there is some $\varphi < \kappa$ such that $|N \cap M^\varphi| = \mu$, since $\mathcal{M}$ is MAD family. It is straightforward that $\langle \alpha, \varphi \rangle \in F'$ since $U(\langle \alpha, \varphi \rangle, \eta) \cap F \neq \emptyset$ for all $\eta < \mu$. $\square$

Corollary 3.5.3. *If $cf(\lambda) \geq \mu$ then a closed unbounded subspace $F \subseteq X$ is not a linearly D-subspace of X. Hence $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is not a linearly D-space.*

Proof. Let $F \subseteq X$ be closed unbounded. $|\pi(D)| < \mu$ for every closed discrete $D \subseteq X$ by Claim 3.5.2. Thus there is no big closed discrete set for the open cover $\{X_{\leq\alpha} : \alpha < \lambda\}$ which shows that F is not linearly D by Theorem 1.2.4. $\square$

Our aim now is to prove that in certain cases the space $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is an aD-space, equivalently every closed subspace of it is irreducible.

Claim 3.5.4. *Every closed, bounded subspace $F \subseteq X$ is a D-subspace of X; hence F is irreducible.*

Proof. Since property D is inherited by closed subspaces, it suffices to prove that $F = X_{\leq\alpha} = \cup\{X_\beta : \beta \leq \alpha\}$ is a D-space.

We do this by induction on $\alpha < \lambda$. Let $U : F \to \tau$ be an ONA. If α is a successor (or $\alpha = 0$), then $F_0 = F \setminus U(\langle \alpha, 0\rangle)$ is closed and $\sup(F_0) < \alpha$ thus we are easily done by induction.

Let $\alpha \in S^\lambda_{\mu'}$ where $\mu \leq \mu' < \lambda$. Then $\sup \pi(F_0) < \alpha$ where $F_0 = F \setminus \cup U[X_\alpha \cap F]$ by Claim 3.5.2. Thus we are easily done by induction and the fact that X_α is closed discrete.

Now let $\nu = cf(\alpha) < \mu$, let $\sup\{\alpha_\xi : \xi < \nu\} = \alpha$ such that $\alpha_0 = 0$ and $\{\alpha_\xi : \xi < \nu\}$ is strictly increasing. Let $F^\xi = \bigcup\{X_\gamma : \alpha_\xi \leq \gamma \leq \alpha_{\xi+1}\}$ if $\xi < \nu$ is limit or $\xi = 0$ and $F^\xi = \bigcup\{X_\gamma : \alpha_\xi < \gamma \leq \alpha_{\xi+1}\}$ if $\xi < \nu$ is a successor. Let $F^\nu = X_\alpha$. Clearly $\{F^\xi : \xi \leq \nu\}$ is a discrete family of disjoint clopen sets such that $\bigcup\{F^\xi : \xi \leq \nu\} = X_{\leq\alpha}$. By induction, for all $\xi < \nu$ there is some closed discrete kernel $D^\xi \subseteq F^\xi$ for the restriction of U to F^ξ. Let $D^\nu = F^\nu$. Then $D = \bigcup\{D^\xi : \xi \leq \nu\}$ is closed discrete and $X_{\leq\alpha} \subseteq \cup U[D]$.

$\square$

To handle the unbounded closed subsets we need the following definition.

Definition 3.5.5. *Let $F_\alpha = F \cap X_\alpha$ for $F \subseteq X$ and $\alpha < \lambda$. A* subset $F \subseteq X$ is high enough *if*

$$|\{\alpha < \lambda : |F_\alpha| = |F|\}| \geq \mu.$$

We say that a subset $F \subseteq X$ is high *if every closed unbounded subset of F is high enough.*

The following rather technical claim will be useful.

Claim 3.5.6. *For any $F \subseteq X$ and ONA $U : F \to \tau$ such that $U(x)$ is a basic open neighborhood of $x \in F$, let*

$$Y_F = \{x \in F : \exists \alpha < \lambda : F_\alpha \subseteq U(x), |F_\alpha| = |F|\},$$

$$\Gamma_F = \{\alpha < \lambda : |F_\alpha| = |F|, \exists x \in F : F_\alpha \subseteq U(x)\}.$$

If F is closed and high enough then $Y_F, \Gamma_F \neq \emptyset$.

Proof. Since $Y_F \neq \emptyset$ iff $\Gamma_F \neq \emptyset$, it is enough to show that there is some $x \in Y_F$. Since F is high enough, $|Z| \geq \mu$ for $Z = \{\alpha' < \lambda : |F| = |F_{\alpha'}|\}$. Let $D = \bigcup\{F_{\alpha'} : \alpha' \in Z\} \subseteq F$. Let $\beta \in S^\lambda_\mu$ be an accumulation point of $Z = \pi(D)$. Then by Claim 3.5.2 there is some $x \in D' \cap X_\beta$ thus $x \in F$. Clearly $x \in Y_F$. □

Theorem 3.5.7. *If the closed unbounded $F \subseteq X$ is high then F is irreducible.*

Proof. Suppose that $\mathcal{U}$ is an open cover of F. We can suppose that we refined it to the form $\{U(x) : x \in F\}$ where each $U(x)$ is basic open. From Claim 3.5.6 we know that $Y_F, \Gamma_F \neq \emptyset$. We define $Y^\xi \subseteq F$ by induction.

- Let $\alpha_0 \in \Gamma_F$ and $Y^0 = \{x \in Y_F : F_{\alpha_0} \subseteq U(x)\}$. Fix some $h^0 : Y^0 \to F_{\alpha_0}$ injection; this exists because $|F_{\alpha_0}| = |F| \geq |Y_F| \geq |Y^0|$.
- Suppose we defined $\alpha_\zeta < \lambda$ and Y^ζ for $\zeta < \xi$. Let

  $$F^\xi = F \setminus \left(\bigcup\{U(x) : x \in \cup\{Y^\zeta : \zeta < \xi\}\} \cup X_{\leq\alpha}\right)$$

  where $\alpha = \sup\{\alpha_\zeta : \zeta < \xi\}$.
- If F^ξ is bounded then stop. Notice that F_ξ is bounded iff $F \setminus \bigcup\{U(x) : x \in \cup\{Y^\zeta : \zeta < \xi\}\}$ is bounded.
- Suppose F^ξ is unbounded. $F^\xi \subseteq F$ is closed too. Thus F^ξ is high enough since F is high. Hence $Y_{F^\xi}, \Gamma_{F^\xi} \neq \emptyset$.
- Let $\alpha_\xi \in \Gamma_{F^\xi}$; thus $|F^\xi_{\alpha_\xi}| = |F^\xi|$ and $F^\xi_{\alpha_\xi}$ is covered by some $U(x)$ for $x \in F^\xi$. Let $Y^\xi = \{x \in Y_{F^\xi} : F^\xi_{\alpha_\xi} \subseteq U(x)\}$. Fix some $h^\xi : Y^\xi \to F^\xi_{\alpha_\xi}$ injection; this exists because $|F^\xi_{\alpha_\xi}| = |F^\xi| \geq |Y_{F^\xi}| \geq |Y^\xi|$.

Lemma 3.5.8. *The induction stops before μ many steps.*

Proof. Suppose we defined this way $\{\alpha_\xi : \xi < \mu\}$ and let $\alpha = \sup\{\alpha_\xi : \xi < \mu\} \in S^\lambda_\mu$. Let $D = \bigcup\{F_{\alpha_\xi} : \xi < \mu\}$. By Claim 3.5.2 there is some $x \in D' \cap X_\alpha$, thus $x \in F$ as well. Clearly $F_{\alpha_\xi} \subseteq U(x)$ for μ many $\xi < \mu$. By the definition of the induction

$(*)$ for every $\zeta < \xi < \mu$ and every $y \in Y^\zeta$: $F^\xi_{\alpha_\xi} \cap U(y) = \emptyset$

Clearly by $(*)$, $x \notin Y^\zeta$ for all $\zeta < \mu$ since there is $\zeta < \xi < \mu$ such that $F^\xi_{\alpha_\xi} \subseteq U(x)$. Moreover $x \notin U(y)$ for every $y \in Y^\zeta$ and $\zeta < \mu$; if $x \in U(y)$ then since $x \neq y$ there is some $\beta < \alpha$ such that $\bigcup\{X_\gamma : \beta < \gamma \leq \alpha\} \subseteq U(y)$. This contradicts $(*)$ since there is $\zeta < \xi < \mu$ such that $\beta < \alpha_\xi$, thus $F^\xi_{\alpha_\xi} \subseteq U(y)$. Thus $x \in F^\xi$ for all $\xi < \mu$. Then $x \in Y^\xi$ for all $\xi < \mu$ such that $F_{\alpha_\xi} \subseteq U(x)$. This is a contradiction. □

Thus let us suppose that the induction stopped at step $\xi < \mu$, meaning that $\widetilde{F} = F \setminus \bigcup\{U(x) : x \in Y\}$ is bounded where $Y = \cup\{Y^\zeta : \zeta < \xi\}$. Let $h = \bigcup\{h^\zeta : \zeta < \xi\}$, $h : Y \to F$ is a 1-1 function since the sets $\mathrm{dom}(h^\zeta) = Y^\zeta$ and $\mathrm{ran}(h^\zeta) \subseteq F^\zeta_{\alpha_\zeta}$ are pairwise disjoint for $\zeta < \xi$. Note that $\mathrm{ran}(h) \subseteq \bigcup\{F_{\alpha_\zeta} : \zeta < \xi\}$ is closed discrete by Claim 3.4.2. For $x \in Y$ let

$$U_0(x) = (U(x) \setminus \mathrm{ran}(h)) \cup \{h(x)\},$$

note that $U_0(x)$ is open. Then

$$\bigcup\{U_0(x) : x \in Y\} = \bigcup\{U(x) : x \in Y\}$$

is a minimal open refinement, since $h(x)$ is only covered by $U_0(x)$ for all $x \in Y$. Let $\mathcal{U}_0 = \{U_0(x) : x \in Y\}$

Let $V(x) = U(x) \setminus \bigcup\{F_{\alpha_\zeta} : \zeta < \xi\}$. Then $\mathcal{V} = \{V(x) : x \in \widetilde{F}\}$ is an open cover of $\widetilde{F}$, refining $\mathcal{U}$; $F_{\alpha_\zeta} \cap \widetilde{F} = \emptyset$ by construction for all $\zeta < \xi$. $\widetilde{F}$ is closed and bounded thus irreducible by Claim 3.5.4, hence there is an irreducible open refinement $\mathcal{V}_0$ of $\mathcal{V}$. It is straightforward that $\mathcal{V}_0 \cup \mathcal{U}_0$ is a minimal open refinement of $\mathcal{U}$ covering F. □

Corollary 3.5.9. *Suppose that $\lambda > \mu = cf(\mu)$ are infinite cardinals such that $cf(\lambda) \geq \mu$. Let $\mathcal{M} = \{M^\varphi : \varphi < \kappa\} \subseteq [\mu]^\mu$ be a MAD family and $\underline{C}$ an S^λ_μ-club sequence. If $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high then $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is a 0-dimensional, Hausdorff, scattered space which is aD however not linearly D.*

Proof. $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is 0-dimensional, Hausdorff, and scattered by Claim 3.4.2 and not linearly D by Corollary 3.5.3. It suffices to show that every closed $F \subseteq X$ is irreducible. If F is bounded then F is a D-space by Claim 3.5.4 hence irreducible. If F is unbounded then F is high since X is high. Hence F is irreducible by Theorem 3.5.7. ☐

3.6 Examples of aD, non linearly D-spaces

In this section we give examples of aD, non linearly D-spaces of the form $X = X[\lambda, \mu, \mathcal{M}, \underline{C}]$. First let us make an observation.

Claim 3.6.1. *If $C_\alpha \subseteq \pi(F)'$ for a closed $F \subseteq X$ and $\alpha \in S^\lambda_\mu$ then $F_\alpha = X_\alpha$.*

Proof. Clearly $\bigcup\{X_\gamma : \gamma \in I^\xi_\alpha\} \cap F \neq \emptyset$ for all $\xi < \mu$. Thus every point in X_α is an accumulation point of F, thus $F_\alpha = X_\alpha$ since F is closed. ☐

Corollaries 3.6.3 and 3.6.5 below give certain examples of high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ spaces.

Proposition 3.6.2. *Suppose that μ is a regular cardinal, $cf(\lambda) \geq \mu^{++}$. Let $\underline{C}$ be an S^λ_μ-club guessing sequence from Theorem 3.3.1. If $\mathcal{M} \subseteq [\mu]^\mu$ is a MAD family of size at least λ then $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ is high.*

Proof. Let $F \subseteq X$ be closed and unbounded. Then $\pi(F)'$ is a club in λ, hence there exists a stationary $S \subseteq S^\lambda_\mu$ such that $C_\alpha \subseteq \pi(F)'$ for all $\alpha \in S$. Thus $F_\alpha = X_\alpha$ by Claim 3.6.1 hence $|F_\alpha| = |\mathcal{M}| = |X|$ for all $\alpha \in S$. ☐

Corollary 3.6.3.

1. *Suppose that $2^\omega \geq \omega_2$. Let $\mathcal{M}$ be a MAD family on ω of size 2^ω and let $\underline{C}$ be an $S^{\omega_2}_\omega$-club guessing sequence from Theorem 3.3.1. Then $X[\omega_2, \omega, \mathcal{M}, \underline{C}]$ is high.*

2. *Suppose that $2^\omega = \omega_1$ and $2^{\omega_1} \geq \omega_3$. Let $\mathcal{M}$ be a MAD family on ω_1 of size 2^{ω_1} (exists by Claim 3.2.2) and let $\underline{C}$ be an $S^{\omega_3}_{\omega_1}$-club guessing sequence from Theorem 3.3.1. Then $X[\omega_3, \omega_1, \mathcal{M}, \underline{C}]$ is high.*

Proposition 3.6.4. *Suppose that $\lambda = \mu^+ > \mu = cf(\mu) > \omega$ and let $\underline{C}$ be an $S^{\mu^+}_\mu$-club guessing sequence from Theorem 3.3.2. If there is a non-stationary MAD family $\mathcal{M}_{NS} \subseteq [\mu]^\mu$ such that $|\mathcal{M}_{NS}| = \mu^+$ then $X = X[\mu^+, \mu, \mathcal{M}_{NS}, \underline{C}]$ is high.*

Proof. Let $\mathcal{M}_{NS} = \{M^\varphi : \varphi < \mu^+\}$ and $\underline{C} = \langle C_\alpha : \alpha \in S^{\mu^+}_\mu \rangle$ such that $C_\alpha = \{a^\xi_\alpha : \xi < \mu\} \subseteq \alpha$. Suppose that the closed $F \subseteq X$ is unbounded. Then $\pi(F)'$ is a club in μ^+, hence there exists a stationary $S \subseteq S^{\mu^+}_\mu$ such that

$$N_\alpha = \{\xi < \mu : a^{\xi+1}_\alpha \in \pi(F)'\} \text{ is stationary in } \mu$$

for all $\alpha \in S$. Fix any $\alpha \in S$, we prove that $|F_\alpha| = |F|$. N_α is stationary so by applying Claim 3.2.1 we get that $|\Phi_\alpha| = \mu^+$ for $\Phi_\alpha = \{\varphi < \mu^+ : |N_\alpha \cap M^\varphi| = \mu\}$. Note that $F \cap \bigcup\{X_\gamma : \gamma \in I^\xi_\alpha\} \neq \emptyset$ for $\xi \in N_\alpha$. Thus $\langle \alpha, \varphi \rangle$ is an accumulation point of F for $\varphi \in \Phi_\alpha$, hence $\{\alpha\} \times \Phi_\alpha \subseteq F_\alpha$. Thus $|F_\alpha| = \mu^+ = |X|$. □

Corollary 3.6.5. *Suppose that $2^{\omega_1} = \omega_2$. Let $\underline{C}$ be an $S^{\omega_2}_{\omega_1}$-club guessing sequence from Theorem 3.3.2 and let $\mathcal{M}_{NS}$ be a nonstationary MAD family on ω_1. Then $X[\omega_2, \omega_1, \mathcal{M}_{NS}, \underline{C}]$ is high.*

Thus, by all means we can deduce the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Note that in any model of ZFC, either $(2^\omega \geq \omega_2)$ or $(2^\omega = \omega_1 \wedge 2^{\omega_1} \geq \omega_3)$ or $(2^{\omega_1} = \omega_2)$. Using Corollaries 3.6.3 and 3.6.5 above, depending on the sizes of 2^ω and 2^{ω_1}, we see that there exists a high $X[\lambda, \mu, \mathcal{M}, \underline{C}]$ space. We are done by Corollary 3.5.9. □

3.7 Consistently on property D and aD

Our main goal in this section is to construct a "locally nice" space which is not linearly D, however every closed subset of it is irreducible; hence aD by Theorem 1.3.4. Then we deduce that such a space, with size less than continuum, cannot exist in ZFC.

3.7.1 Preliminaries

We will use the following set-theoretical assumption:

($\diamondsuit^*$) there is a $\diamondsuit^*$-*sequence*, meaning that there exists an $\{\mathcal{A}_\alpha : \alpha \in \lim(\omega_1)\}$ such that $\mathcal{A}_\alpha \subseteq [\alpha]^\omega$ is countable and for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ such that $X \cap \alpha \in \mathcal{A}_\alpha$ for all $\alpha \in C$.

Also, we need the following easy claim about MAD families.

Claim 3.7.1. *If $\{N_i : i \in \omega\} \subseteq [\omega]^\omega$ then there is a MAD family $\mathcal{M} \subseteq [\omega]^\omega$ of size 2^ω such that for all $M \in \mathcal{M}$ and $i \in \omega$: $|M \cap N_i| = \omega$.*

Proof. We will construct the MAD family $\mathcal{M}$ on $\mathbb{Q}$. We can suppose that each N_i is dense in $\mathbb{Q}$. Let $\mathbb{R} = \{x_\alpha : \alpha < 2^\omega\}$ and for all $\alpha < 2^\omega$ let $S_\alpha \subseteq \mathbb{Q}$ such that S_α is a convergent sequence with limit point x_α and $|S_\alpha \cap N_i| = \omega$ for all $i \in \omega$. Then $\mathcal{S} = \{S_\alpha : \alpha < 2^\omega\}$ is almost disjoint, let $\mathcal{T} = \{T_\alpha : \alpha < \lambda\} \subseteq [\mathbb{Q}]^\omega$ such that $\mathcal{S} \cup \mathcal{T}$ is MAD. Then $\mathcal{M} = \{S_\alpha \cup T_\alpha : \alpha < \lambda\} \cup \{S_\alpha : \lambda \leq \alpha < 2^\omega\}$ is a MAD family with the desired property. $\square$

The following result of Zoltán Balogh will play a key role in proving our independence result.

Definition 3.7.2. *A space X is said to be* locally nice *iff X is locally countable and locally compact.*

Let us note that every locally nice space is 0-dimensional, Tychonoff and first-countable.

Theorem 3.7.3 ([5, Theorem 2.2])**.** *Suppose MA. Then for any locally nice space X of cardinality $< 2^\omega$ exactly one of the following is true:*

- *X is the countable union of closed discrete subspaces,*
- *X contains a perfect preimage of ω_1 with the order topology.*

Let us state a final claim, which will be used later.

Claim 3.7.4. *(i) If the space F is a perfect preimage of ω_1 then F is countably compact, non compact.*

(ii) If X is first-countable and $F \subseteq X$ is a perfect preimage of ω_1 then F is closed in X.

Proof. (i) It is known that under perfect mappings, the preimage of a compact space is compact (see [14, Theorem 3.7.2]). Take any countably infinite $A \subseteq F$ and perfect surjection $f : F \to \omega_1$. There is some $\alpha < \omega_1$ such that $f[A] \subseteq \alpha + 1$. Thus A is the subset of the compact set $f^{-1}[\alpha + 1]$. (ii) is a consequence of (i). $\square$

The following can be easily seen now.

Corollary 3.7.5. *Suppose that X is a first-countable space which is aD or linearly D. Then X does not contain a perfect preimage of ω_1.*

3.7.2 The main result

Theorem 3.7.6. *Suppose* ($\diamondsuit^*$). *There is a locally nice, 0-dimensional T_2 space X of size ω_1 such that X is not linearly D, however every closed subset $F \subseteq X$ is irreducible; equivalently X is an aD-space.*

Proof. We will define a topology on $X = \omega_1 \times \omega_1$. Let $X_\alpha = \{\alpha\} \times \omega_1$ and $X_{<\alpha} = \alpha \times \omega_1$ for $\alpha < \omega_1$.

Definition 3.7.7. *The set $A \in [X]^\omega$* runs up to *$\alpha < \omega_1$ iff $A = \{(\alpha_n, \beta_n) : n \in \omega)\} \subseteq X_{<\alpha}$ such that $\alpha_0 \leq ... \leq \alpha_n \leq ...$ and $\sup\{\alpha_n : n \in \omega\} = \alpha$.*

Note that if $A \subseteq X$ runs up to some $\alpha < \omega_1$ then $A \cap X_\beta$ is finite for all $\beta < \omega_1$.

We need the following consequence of ($\diamondsuit^*$). Let $\pi(A) = \{\alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset\}$ for $A \subseteq X$.

Claim 3.7.8. ($\diamondsuit^*$) *There exists a sequence $\{A_\alpha : \alpha \in \lim(\omega_1)\} \subseteq [X]^\omega$ with $A_\alpha = \bigcup\{A^n_\alpha : n \in \omega\}$ for all $\alpha \in \lim(\omega_1)$ such that*

1. *$|A^n_\alpha| = \omega$ for all $n \in \omega$,*
2. *A_α runs up to α,*
3. *for all $Y \subseteq X$ if $|\pi(Y)| = \omega_1$ then*

$$\exists \text{ club } C \subseteq \omega_1 \text{ such that } \forall \alpha \in C \exists n \in \omega (A^n_\alpha \subseteq Y).$$

Proof. Let $\{\mathcal{A}_\alpha : \alpha \in \lim(\omega_1)\}$ denote a $\diamondsuit^*$-sequence. Let $i : \omega_1 \times \omega_1 \to \omega_1$ denote a bijection which maps $\big((\alpha+1)\times(\alpha+1)\big) \setminus (\alpha \times \alpha)$ to $\omega \cdot (\alpha+1) \setminus \omega \cdot \alpha$. Let

$$\widetilde{\mathcal{A}}_\alpha = \{i^{-1}(A) : A \in \mathcal{A}_{\omega\cdot\alpha}, \sup\big(\pi(i^{-1}(A))\big) = \alpha\}$$

and let $A_\alpha = \bigcup\{A^n_\alpha : n \in \omega\}$ such that

1. $|A^n_\alpha| = \omega$ for all $n \in \omega$,
2. A_α runs up to α,

(3)′ for all $B \in \widetilde{\mathcal{A}}_\alpha$ there is $n \in \omega$ such that $A^n_\alpha \subseteq B$, for all $\alpha \in \lim(\omega_1)$. We claim that the sequence $\{A_\alpha : \alpha \in \lim(\omega_1)\}$ has the desired properties. Let $Y \subseteq X$ such that $|\pi(Y)| = \omega_1$. There is some club $C_0 \subseteq \omega_1$ such that $Y \cap X_{<\alpha} \subseteq \alpha \times \alpha$ for $\alpha \in C_0$. There is some club $C_1 \subseteq \omega_1$ such that $\alpha \cap i[Y] \in \mathcal{A}_\alpha$ for $\alpha \in C_1$. Let $C_2 = \{\alpha < \omega_1 : \omega \cdot \alpha \in C_1\}$; clearly, C_2 is a club. Let $C = C_0 \cap C_2 \cap \pi(Y)'$. Fix some $\alpha \in C$. Then $\omega \cdot \alpha \cap i[Y] = A$ for some $A \in \mathcal{A}_{\omega \cdot \alpha}$, thus $i[Y \cap X_{<\alpha}] = A$ since $\omega \cdot \alpha = i[\alpha \times \alpha]$ and $Y \cap X_{<\alpha} \subseteq \alpha \times \alpha$. Hence $i^{-1}(A) = Y \cap X_{<\alpha}$ and $i^{-1}(A) \in \widetilde{\mathcal{A}}_\alpha$ because $\alpha \in \pi(Y)'$. Thus there is $n \in \omega$ such that $A_\alpha^n \subseteq Y$ by (3)$'$. $\square$

Let $\{A_\alpha : \alpha \in \lim(\omega_1)\} \subseteq [X]^\omega$ denote a sequence with $A_\alpha = \bigcup\{A_\alpha^n : n \in \omega\}$ for $\alpha \in \lim(\omega_1)$ from Claim 3.7.8. We want to define the topology on X such that

- X_α is closed discrete for all $\alpha < \omega_1$,
- $X_{<\alpha}$ is open for all $\alpha \in \omega_1$,
- if $A \in [X]^\omega$ runs up to α then A has an accumulation point in X_α,
- $X_\alpha \subseteq \overline{A_\alpha^n}$ for all $\alpha \in \lim(\omega_1)$ and $n \in \omega$.

Let $\mathcal{M}_\alpha \subseteq [A_\alpha]^\omega$ denote a MAD family on A_α for $\alpha \in \lim(\omega_1)$ such that $|M \cap A_\alpha^n| = \omega$ for all $M \in \mathcal{M}_\alpha$ and $n \in \omega$; such an $\mathcal{M}_\alpha$ exists by Claim 3.7.1. Enumerate $\mathcal{M}_\alpha = \{M_\alpha^\beta : \beta < \omega_1\}$.

We define topologies $\tau_{<\alpha}$ on $X_{<\alpha}$ by induction on $\alpha < \omega_1$ such that $\tau_{<\alpha} \cap \mathcal{P}(X_{<\beta}) = \tau_{<\beta}$ for all $\beta < \alpha < \omega_1$. This way we will get a topology τ on X if we take $\cup\{\tau_{<\alpha} : \alpha < \omega_1\}$ as a base.

Suppose $\alpha < \omega_1$ and we have defined the topology $(X_{<\alpha}, \tau_{<\alpha})$ such that

(i) $(X_{<\alpha}, \tau_{<\alpha})$ is a locally countable, locally compact, 0-dimensional T_2 space,

(ii) for all $\alpha' < \alpha$ and $x \in X_{\alpha'}$ there is some neighborhood G of x such that $G \cap X_{\alpha'} = \{x\}$,

(iii) $(\alpha_0, \alpha_1] \times \omega_1 \subseteq X_{<\alpha}$ is clopen for all $\alpha_0 < \alpha_1 < \alpha$.

If $\alpha \in \omega_1 \setminus \lim(\omega_1)$ then let X_α be discrete. Suppose $\alpha \in \lim(\omega_1)$ and let us enumerate $\{F \subseteq X_{<\alpha} \setminus A_\alpha : F \text{ runs up to } \alpha\}$ as $\{F_\alpha^\beta : \beta < \omega_1\}$.

Definition 3.7.9. *A subspace $A \subseteq T$ of a topological space T is* completely discrete *iff there is a discrete family of open sets $\{G_a : a \in A\}$ such that $a \in G_a$ for all $a \in A$.*

The following claim will be useful later.

Claim 3.7.10. *Suppose that $A = \{(\alpha_n, \beta_n) : n \in \omega\} \subseteq X$ runs up to α. Then A is completely discrete in $X_{<\alpha}$; hence closed discrete.*

Proof. Let $G_0 = (0, \alpha_0] \times \omega_1$ and $G_{n+1} = (\alpha_n, \alpha_{n+1}] \times \omega_1$ for $n \in \omega$. G_n is open for all $n \in \omega$ by inductional hypothesis (iii). Note that $\{G_n : n \in \omega\}$ is a discrete family of open sets such that $A \cap G_n$ is finite for all $n \in \omega$. Let $\mathcal{G}_n$ denote a finite, disjoint family of clopen subsets of G_n such that for all $a \in A \cap G_n$ there is exactly one $G \in \mathcal{G}_n$ such that $a \in G$. Then the discrete family $\cup\{\mathcal{G}_n : n \in \omega\}$ shows that A is completely discrete. □

In step $\alpha \in \lim(\omega_1)$ we define the neighborhoods of points in $X_\alpha = \{(\alpha, \beta) : \beta < \omega_1\}$ by induction on $\beta < \omega_1$ such that:

(a) $X_{<\alpha} \cup \{(\alpha, \beta') : \beta' \leq \beta\}$ is locally countable, locally compact and 0-dimensional T_2,

(b) there is some neighborhood U of (α, β) such that $U \cap A_\alpha \subseteq M^\beta_\alpha$,

(c) M^β_α converges to (α, β),

(d) F^β_α accumulates to (α, β') for some $\beta' \leq \beta$.

We need the following lemma to carry out the induction on $\beta < \omega_1$.

Lemma 3.7.11. *Suppose that $(T \cup S, \tau)$ is a locally countable, locally compact and 0-dimensional T_2 space such that T is open and S is countable. Let $D = \{d_n : n \in \omega\} \subseteq T$ closed discrete in $T \cup S$ and completely discrete in T. Let $r \notin T \cup S$. Then there is a topology ρ on $R = T \cup S \cup \{r\}$ such that*

- *(R, ρ) is locally countable, locally compact and 0-dimensional T_2,*
- *$\rho|_{(T \cup S)} = \tau$,*
- *D converges to r and $r \notin \overline{S}$ in (R, ρ).*

Proof. Suppose that $d_n \in G_n$ such that $\{G_n : n \in \omega\}$ is a family of open sets which is discrete in T. For each $n \in \omega$ let $\{B^n_i : i \in \omega\}$ denote a neighborhood base of d_n such that

- $G_n \supseteq B^n_0 \supseteq B^n_1 \supseteq ...$ and
- B^n_i is countable, compact and clopen for all $n, i \in \omega$.

Since $S \cap D = \emptyset$ there is some clopen neighborhood U_s of each $s \in S$ such that $U_s \cap D = \emptyset$. There is $g_s : \omega \to \omega$ such that

$$U_s \cap B^n_{g_s(n)} = \emptyset \text{ for all } n \in \omega.$$

Since S is countable, there is $g : \omega \to \omega$ such that for all $s \in S$ there is some $N \in \omega$ such that $g_s(n) \leq g(n)$ for all $n \geq N$. Define the topology ρ on R as follows. Let

$$B_N = \{r\} \cup \bigcup \{B^n_{g(n)} : n \geq N\} \text{ and } \mathcal{B} = \{B_N : N \in \omega\}.$$

Let ρ be the topology on R generated by $\tau \cup \mathcal{B}$.

Clearly $\rho|_{(T \cup S)} = \tau$. We claim that (R, ρ) is locally countable, locally compact and 0-dimensional. Since $\mathcal{B}$ is a neighborhood base for r, it suffices to prove that each $B \in \mathcal{B}$ is countable, compact (trivial) and clopen. Let $N \in \omega$ then B_N is clopen in T since $\bigcup \{B^n_{g(n)} : n \in \omega\}$ is a family of clopen sets which is discrete in T guaranteed by the discrete family $\{G_n : n \in \omega\}$. Let $s \in S$. There is $N \in \omega$ such that $U_s \cap B^n_{g(n)} = \emptyset$ for $n \geq N$. There is some neighborhood $V \in \tau$ of s such that $V \cap \bigcup \{B^n_{g(n)} : n < N\} = \emptyset$ since s is not in the closed set $\bigcup \{B^n_{g(n)} : n < N\}$. Thus $(U_s \cap V) \cap B_N = \emptyset$. This proves that B_N is clopen.

We claim that (R, ρ) is T_2. Let $s \in S$, then there is $N \in \omega$ such that $U_s \cap B^n_{g(n)} = \emptyset$ for $n \geq N$, thus $B_N \cap U_s = \emptyset$. As noted before $B_N \cap T$ is closed and clearly $\bigcap \{B_N \cap T : N \in \omega\} = \emptyset$. This yields that any point $t \in T$ and r can be separated, thus (R, ρ) is T_2.

Clearly D converges to r and $S \cap B = \emptyset$ for any $B \in \mathcal{B}$ thus $r \notin \overline{S}$. □

Suppose we are in step $\beta < \omega_1$ and we defined the neighborhoods of points in $X_{<\alpha} \cup \{(\alpha, \beta') : \beta' < \beta\}$. We use Lemma 3.7.11 to define the neighborhoods of $r = (\alpha, \beta)$. Let $T = X_{<\alpha}$ and $S = \{(\alpha, \beta') : \beta' < \beta\} \cup (A_\alpha \setminus M^\beta_\alpha)$. Note that $F^\beta_\alpha \cup M^\beta_\alpha$ runs up to α thus closed and completely discrete in T by Claim 3.7.10. Also, M^β_α is closed discrete in $T \cup S$ by inductional hypothesis (b) for (α, β') where $\beta' < \beta$.

- If F^β_α accumulates to $x_{\beta'}$ for some $\beta' < \beta$ then let $D = M^\beta_\alpha$.
- If F^β_α is closed discrete in $T \cup S$ then let $D = M^\beta_\alpha \cup F^\beta_\alpha$.

Note that D is closed discrete in $T \cup S$. By Claim 3.7.11 we can define the neighborhoods of $r = (\alpha, \beta)$ such that the resulting space satisfies conditions (a), (b),(c) and (d). After carrying out the induction on β, the resulting topology on X_α clearly satisfies conditions (i),(ii) and (iii). This completes the induction.

As a base, the family $\bigcup\{\tau_{<\alpha} : \alpha \in \lim(\omega_1)\}$ generates a topology τ on X which is locally countable, locally compact and 0-dimensional T_2. Observe that X_α is closed discrete and $X_{<\alpha}$ is open for all $\alpha < \omega_1$ (by inductional hypothesises (ii) and (iii)) .

Claim 3.7.12. *Suppose that $F \subseteq X$ runs up to some $\alpha \in \lim(\omega_1)$. Then there is some $\beta < \omega_1$ such that F accumulates to (α, β). Equivalently, if $G \subseteq X$ is open and $X_\alpha \subseteq G$ then there is some $\alpha' < \alpha$ such that $(\alpha', \alpha] \times \omega_1 \subseteq G$.*

Proof. There is some $\beta < \omega_1$ such that $F = F^\beta_\alpha$. Thus by inductional hypothesis (d) there is some $\beta' \leq \beta$ such that F accumulates to (α, β'). □

Claim 3.7.13. *X is not linearly D.*

Proof. If $D \subseteq X$ is closed discrete then $\pi(D)$ is finite by Claim 3.7.12. Thus there is no big closed discrete set for the cover $\{X_{<\alpha} : \alpha < \omega_1\}$. □

Our next aim is to prove that all closed subspaces of X are irreducible.

Claim 3.7.14. *If $|\pi(F)| = \omega$ for a closed $F \subseteq X$ then F is a D-space, hence irreducible.*

Proof. Since $F = \cup\{F \cap X_\alpha : \alpha \in \pi(F)\}$ is a countable union of closed discrete sets, F is a D-space by Proposition 1.1.2. We mention that if the ONA U on F has closed discrete kernel D then we get an irreducible cover by taking the following open refinement: $\{(U(d) \setminus D) \cup \{d\} : d \in D\}$. □

Claim 3.7.15. *If $|\pi(A)| = \omega_1$ for $A \subseteq X$ then there is a club $C \subseteq \omega_1$ such that $C \times \omega_1 \subseteq A'$. As a consequence, if $\pi(U)$ is stationary for the open $U \subseteq X$ then there is some $\alpha < \omega_1$ such that $X \setminus U \subseteq \alpha \times \omega_1$.*

Proof. There is a club $C \subseteq \omega_1$ by Claim 3.7.8 such that for all $\alpha \in C$ there is $n \in \omega$ such that $A^n_\alpha \subseteq A$. We will prove that $X_\alpha \subseteq A'$ for $\alpha \in C$. Take any point $(\alpha, \beta) \in X_\alpha$. $|M^\beta_\alpha \cap A^n_\alpha| = \omega$ for all $\beta < \omega_1$ by the construction of the MAD family $\mathcal{M}_\alpha$ and M^β_α converges to (α, β) by inductional hypothesis (c). Thus A^n_α accumulates to (α, β), hence $X_\alpha \subseteq A'$. □

Claim 3.7.16. *If $|\pi(F)| = \omega_1$ for a closed $F \subseteq X$ then F is irreducible.*

Proof. Take an open cover of F, say $\mathcal{U}$. We can suppose that we refined it to the form $\mathcal{U} = \{U(x) : x \in F\}$, where $U(x)$ is a neighborhood of $x \in F$. From Claim 3.7.15 we know that there is some club $C \subseteq \omega_1$ such that $C \times \omega_1 \subseteq F$. For $\alpha \in C$ define the open set $G_\alpha = \cup\{U(x) : x \in X_\alpha\}$. For every $\alpha \in C$ there is some $\delta(\alpha) < \alpha$ such that $(\delta(\alpha), \alpha] \times \omega_1 \subseteq G_\alpha$; by Claim 3.7.12. So there is some $\delta < \omega_1$ and a stationary $S \subseteq C$ such that $(\delta, \alpha] \times \omega_1 \subseteq G_\alpha$ for all $\alpha \in S$. Fix some $\delta_0 > \delta$ such that $X_{\delta_0} \subseteq F$. Let $S_0 = S \setminus (\delta_0 + 1)$. For all $\alpha \in S_0$ there is $d_\alpha \in X_\alpha \subseteq F$ such that $(\delta_0, \alpha) \in U(d_\alpha)$. Let us refine these sets: $U_0(d_\alpha) = \big(U(d_\alpha) \setminus (\{\delta_0\} \times S_0)\big) \cup \{(\delta_0, \alpha)\}$ for all $\alpha \in S_0$; let $\mathcal{U}_0 = \{U_0(d_\alpha) : \alpha \in S_0\}$. Clearly $\mathcal{U}_0$ is an open refinement of $\mathcal{U}$ which is minimal and $\{d_\alpha : \alpha \in \omega_1\} \subseteq \cup\mathcal{U}_0$. Since S_0 is stationary and $S_0 \subseteq \pi[\cup\mathcal{U}_0]$ we get that there is some $\gamma < \omega_1$ such that $F_1 = F \setminus \cup\mathcal{U}_0 \subseteq \gamma \times \omega_1$ by Claim 3.7.15. So by Claim 3.7.14 the closed set F_1 is a D-space, hence irreducible. Take a minimal open refinement of the cover $\{U(x) \setminus (\{\delta_0\} \times S_0) : x \in F_1\}$, let this be $\mathcal{U}_1$. The union $\mathcal{U}_0 \cup \mathcal{U}_1$ is an open refinement of $\mathcal{U}$ which covers F and minimal. □

This proves that all closed subspaces of X are irreducible. Hence X is an aD-space by Theorem 1.3.4.

□

Finally, we can observe the following.

Proposition 3.7.17. *Suppose MA. Let X be a locally nice space of cardinality $< 2^\omega$. Then the following are equivalent:*

(1) X is a D-space,

(2) X is a linearly D-space,

(3) X is an aD-space.

Proof. In each case, X does not contain a perfect preimage of ω_1 by Corollary 3.7.5. Hence, X is σ-closed discrete by Balogh's Theorem 3.7.3 which finishes the proof. □

Thus we can deduce the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. If $\mathrm{MA}_{\aleph_1}$ holds, then every locally nice aD-space of cardinality ω_1 is a D-space by Proposition 3.7.17. If $(\diamondsuit^*)$ holds, then there is a locally nice, 0-dimensional T_2 space X of size ω_1 such that X is not linearly D, but aD by Theorem 3.7.6. This completes the proof. □

However, the following remain open.

Problem 3.7.18. *Is there a ZFC example of a locally nice, T_2 space X such that X is not (linearly) D however aD?*

Bibliography

[1] U. Abraham, M. Magidor, Cardinal Arithmetic – Handbook of Set Theory (Eds. Foreman, Kanamori), Volume 2, 1149-1229.

[2] A.V.Arhangel'skii and R. Buzyakova, Addition theorems and D-spaces, Comment. Mat. Univ. Car. 43(2002), 653-663.

[3] A.V. Arhangel'skii, D-spaces and covering properties, Topology and Appl. 146-147(2005), 437- 449.

[4] L. Aurichi, D-spaces, topological games, and selection principles, Topology Proc. 36 (2010), 107-122.

[5] Z. Balogh, Locally nice spaces and Martin's Axiom, Comment. Math. Univ. Carolin. 24 (1983) 63-87.

[6] J. E. Baumgartner, Almost-disjoint sets, the dense set problem and the partition calculus, Ann. Math. Logic 9 (1976), no. 4, 401-439.

[7] C.R. Borges and A. Wehrly, A study of D-spaces, Topology Proc. 16(1991), 7-15.

[8] D. Burke, Weak bases and D-spaces, Comment. Mat. Univ. Car. 48(2007), 281-289.

[9] R. Buzyakova, On D-property of strong Σ-spaces, Comment. Mat. Univ. Car. 43(2002), 493- 495.

[10] P. de Caux, Yet another property of the Sorgenfrey plane, Topology Proc. 6(1981), 31-43.

[11] E.K. van Douwen and W. Pfeffer, Some properties of the Sorgenfrey line and related spaces, Pacific J. Math. 81 (1979), 371â377.

[12] E.K. van Douwen and H. Wicke, A real, weird topology on the reals, Houston J. Math. 3 (1977), no. 1, 141â152.

[13] T. Eisworth, On D-spaces, in: Open Problems in Topology II, E. Pearl, ed., Elsevier, Amsterdam, 2007, 129-134.

[14] R. Engelking, General Topology, Berlin: Heldermann, 1989.

[15] G. Gruenhage, A survey of D-spaces, Contemporary Math. 533(2011), 13-28.

[16] G. Gruenhage, Generalized metric spaces, in the Handbook of Set-theoretic Topology, K. Kunen and J.E. Vaughan, eds., North-Holland, Amsterdam, 1984, 423-501.

[17] H. Guo and H.J.K. Junnila, On D-spaces and their unions, preprint.

[18] H. Guo and H.J.K. Junnila, On spaces which are linearly D, Topology and Appl., Volume 157, Issue 1, 1 January 2010, Pages 102-107.

[19] W. Hurewicz, Uber eine Verallgemeinerung des Borelschen Theorems, Math. Z. 24(1926), 401-421.

[20] M. Hrusšák and J.T. Moore, Twenty problems in set theoretic topology, in: Open Problems in Topology II, E. Pearl, ed., Elsevier, Amsterdam, 2007, 111-114.

[21] K. Kunen, Set Theory, An Introduction to Independence Proofs, Studies in Logic and the Foundations of Mathematics v102, North-Holland, 1983.

[22] E. Pearl, Linearly Lindelöf problems, in: Open Problems in Topology II, E. Pearl, ed., Elsevier, Amsterdam, 2007, 225-231.

[23] D. Repovš and L. Zdomskyy, On the Menger covering property and D-spaces, Proceedings of the American Mathematical Society, to appear

[24] S. Shelah, Cardinal Arithmetic – Oxford Logic Guides, vol. 29, Oxford Univ. Press (1994).

[25] S. Shelah, Colouring and non-productivity of $\aleph_2$-c.c. – Annals Pure and Applied Logic 84 (1997) 153-174.

[26] W. Shi and H. Zhang, A note on D-spaces, preprint 2010.

[27] D. T. Soukup, Properties D and aD are different, Top. Proc. 38 (2011) pp. 279-299.

[28] D. T. Soukup, Constructing aD, non D-space, Topology and its Applications, Volume 158, Issue 10, 15 June 2011, Pages 1219-1225.

[29] D. T. Soukup, L. Soukup, Club guessing for dummies, Arxiv note.

[30] P. Szeptycki, A counterexample in the theory of D-spaces, preprint.

[31] F.D. Tall, Productively Lindelöf spaces may all be D, Canad. Math. Bull. to appear.