Davies-trees in infinite combinatorics

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Logic Colloquium 2014
July 15, 2014, Vienna

Outline - our goals

- point out a typical situation in recursive or inductive constructions,
- introduce **a technique** to deal with these difficulties,
- present old applications,
- present **new applications** and advertise the **handout**.

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A simple example

Theorem (P. Komjáth, 1984)

Suppose that a family of countable sets $\mathcal{X}$ is n-almost disjoint for some $n \in \mathbb{N}$, i.e. $|A \cap B| < n$ for every $A \neq B \in \mathcal{X}$.
Then $\mathcal{X}$ is essentially disjoint, i.e. we can select finite $F_A \subseteq A$ for each $A \in \mathcal{X}$ so that $\{A \setminus F_A : A \in \mathcal{X}\}$ is pairwise disjoint.

List $\mathcal{X}$ and inductively select the finite sets.

- if $\mathcal{X}$ is countable then no worries,
- if $\mathcal{X}$ is uncountable then pack into countable pieces $\{\mathcal{X}_\alpha : \alpha < \kappa\}$,
- if $|(\bigcup \mathcal{X}_{<\alpha}) \cap A| < \omega$ for $A \in \mathcal{X}_\alpha$ and $\alpha < \kappa$
  then we can deal with them separately!

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Elementary submodels and chains

- if V is a model of ZFC (or a large fragment of it) then $M \prec V$ iff

$$M \models \varphi \iff V \models \varphi$$

for every formula φ with parameters from M,

- we use chains of countable elementary submodels to produce the pieces $\mathcal{X}_\alpha$,
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Special sequences of elementary submodels

- if $(M_\alpha)_{\alpha<\omega_1}$ **is a chain** then $\bigcup_{\alpha<\beta} M_\alpha$ **is a el. submodel** as well,
- but we want to deal with structures of **arbitrary size**,
- the idea is to **switch from chains** to special sequences of countable submodels, called **Davies-trees**,
- **Davies-tree** $\approx$ a sequence $(M_\alpha)_{\alpha<\kappa}$ such that $\bigcup_{\alpha<\beta} M_\alpha$ is the **union of finitely many submodels**,
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- (S. Jackson, R. D. Mauldin, 2002) There is a **subset of** $\mathbb{R}^2$ which intersect each isometric copy of $\mathbb{Z} \times \mathbb{Z}$ in **exactly one point**.
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New applications - clouds

Definition

*We call $A \subset \mathbb{R}^2$ a **cloud around a point** a iff every line L through a intersect A in a finite set.*

Theorem (P. Komjáth, 2001)

*The **Continuum Hypothesis** is equivalent to the statement that $\mathbb{R}^2$ **is the union of 3 clouds**.*

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Any questions?