NOTES ON GALVIN'S CONJECTURE

DÁNIEL T. SOUKUP

Abstract. We present a weaker version of a recent result of D. Raghavan and S. Todorcevic that highlights their main combinatorial ideas in proving Galvin's conjecture (assuming the existence of some large cardinals). Using their arguments, we show that if there is a precipitous ideal on ω_1 then for any uncountable set of reals X and finite colouring $c : [X]^2 \to \ell$ there is a homeomorphic copy $Y \subseteq X$ of $\mathbb{Q}$ so that $c \restriction [Y]^2$ assumes at most two colours.

Suppose that $\mathcal{I}$ is a σ-ideal. For the purposes of this note, it is safe to think of $\mathcal{I}$ as the non-stationary ideal on ω_1 or the ideal of meager sets in a Baire space.

We define the following game:

$$\begin{array}{c|cccc} \mathrm{I} & S_0 & & S_2 & \dots \\ \hline \mathrm{II} & & S_1 & & \dots \end{array}$$

where Player I (called Empty) and II (called Non Empty) alternately play $\mathcal{I}$-positive sets forming a decreasing sequence (i.e., $S_{n+1} \subseteq S_n$) in ω steps. Empty wins if $\bigcap_{n\in\omega} S_n = \emptyset$.

Those ideals $\mathcal{I}$ such that Empty has no winning strategy are called *precipitous*. It is equiconsistent with the existence of a measurable cardinal that the non-stationary ideal on ω_1 is precipitous [2]. See Section 8 of M. Foreman's survey [1] on making certain natural ideals (such as the non-stationary, null or meager ideal) precipitous by forcing.

Our goal is to present the following result and its surprisingly elementary proof.

Theorem 0.1. [3] *Suppose that there is a precipitous ideal on ω_1. Then for any uncountable set of reals X and finite ℓ,*

$$X \to (top\ \mathbb{Q})^2_{\ell,2}.$$

Raghavan and Todorcevic provide a detailed and enjoyable presentation of a much stronger result in [3]. In fact, one of their illuminating comments is that the above theorem is an easy modification of their main result. We believe that most of the combinatorial ideas from [3] appear in the arguments we collected here. We will use $\mathcal{I}$ to have a notion of largeness which corresponds to the stationary tower in [3]. The ideal being precipitous is used analogously to [3, Theorem 21]. However, the actual construction of the homeomorphic copy of $\mathbb{Q}$ appear with much less technical detail here. So we hope our notes provide a helpful introduction to understanding the more general setting of [3] which involves a general class of topological spaces (i.e., non left-separated spaces with a point-countable base) and the stationary tower machinery. To this end, we closely followed the notation and proof-structure from [3].

We fix X with a countable base $\mathcal{B}$ for the topology. We can shrink X to have size $\aleph_1$ and assume that the precipitious ideal $\mathcal{I}$ lives on X. For each point $x \in X$, let $(U_{x,n})_{n\in\omega}$ denote

Date: September 25, 2018.

a decreasing neighbourhood base selected from $\mathcal{B}$. We will simply say that a subset of X is *large* (think of being stationary or non-meager) if it is $\mathcal{I}$-positive. *Almost every* will refer to all but ideal many.

Lemma 0.2. *Suppose that $S \subseteq X$ is large.*

1. *For almost every $x \in S$ and all $n \in \omega$, $U_{x,n} \cap S$ is large.*
2. *For any $n \in \omega$, there is a U so that $\{x \in S : U_{x,n} = U\}$ is large.*

From now on, we will only work with large sets S so that $U_{x,n} \cap S$ is large for every $x \in S$. Condition (1) says that this is possible.

Fix the finite colouring $c : [X]^2 \to \ell$ as well. We say that x is i-large in T if $\{y \in T : c(x, y) = i\}$ is large.

Lemma 0.3. *For any $x \in X$ and large $T \subseteq X$, there is an i so that x is i-large in T.*

Definition 0.4. *We say that a pair of large sets S, T is (i, j)-saturated if for any large $S' \subset S$ and $T' \subset T$ both sets*

$$\{x \in S' : x \text{ is } i\text{-large in } T'\}$$

and

$$\{y \in T' : y \text{ is } j\text{-large in } S'\}$$

are large.

Note that being (i, j)-saturated is hereditary to subsets.

Main Lemma 0.5. *There is (i, j) and some large $S_1 \subseteq X$ so that for any large $S \subseteq S_1$ there is an (i, j)-saturated pair $S', T' \subseteq S$.*

Proof. This is a classical exhaustion argument. Suppose that the statement of the lemma fails: then, by shrinking X ℓ^2-many times, we arrive at a set S_1 so that no pair S, T below S_1 is (i, j)-saturated for any (i, j). By unravelling the definition of (i, j)-saturated, we construct large sets $S'_0 \supseteq S'_1 \supseteq \cdots \supseteq S'_{\ell^2}$ and $T'_0 \supseteq T'_1 \supseteq \cdots \supseteq T'_{\ell^2}$ so that for any (i, j) there is some $k < \ell^2$ and sets S^*_k, T^*_k from the ideal such that either

$$S^*_k = \{x \in S'_k : x \text{ is i-large in } T'_k\}$$

or

$$T^*_k = \{y \in T'_k : y \text{ is i-large in } S'_k\}.$$

We simply form $S^* = S'_{\ell^2} \setminus \bigcup_{k<\ell^2} S^*_k$ and $S^* = T'_{\ell^2} \setminus \bigcup_{k<\ell^2} T^*_k$. Now, pick any $x \in S^*$ and i so that x is i-large in T^*. Similarly, let $y \in T^*$ arbitrary and find j so that y is j-large in S^*. However, this should not be possible as we considered (i, j) at some step k and thrown away either all i-large points from the S-side or all j-large points from the T-side. $\square$

Let us fix this pair of colours (i, j) and, to simplify notation, we assume that $X = S_1$ satisfies the previous lemma. The last lemma we need is the following:

Main Lemma 0.6. *For any large $T \subset X$ and for almost every $x \in T$ there is a sequence $(T_n)_{n\in\omega}$ of large sets so that*

1. *$T_n \subseteq T \cap U_{x,n}$,*
2. *$c(x, y) = i$ for all $y \in T_n$, and*
3. *T_k, T_n is (i, j)-saturated for all $n < k < \omega$.*

From this, one can deduce the proof of the main theorem. Moreover, the proof of this lemma is the only place where we use the assumption that $\mathcal{I}$ is precipitous.

Proof. Let us say that *x is an (i,j)-winner in a large set S* if $x \in S$ and there is a sequence S_n, T_n of (i,j)-saturated pairs so that

(1) $S_n, T_n \subset U_{x,n}$,
(2) $S_{n+1}, T_{n+1} \subset S_n$, and
(3) $c(x,y) = i$ for any $y \in T_n$.

It suffices to show the following now.

Lemma 0.7. *For any large S, almost every $x \in S$ is an (i,j)-winner in S.*

Proof. Suppose this is not the case and that there is a large set S' of $x \in S$ which are not (i,j)-winners. Our plan is the following: we will use our assumption to define a strategy for Empty in the game introduces at the beginning, starting with S'. This strategy cannot be winning so there is a play which produces a sequence of large sets with non empty intersection. We will show that any point X in the intersection is an (i,j)-winner in S, contradicting that $x \in S$.

To define the strategy σ, we need the following simple fact.

Lemma 0.8. *For any large S'' and $n \in \omega$ there is an (i,j)-saturated pair $R, T \subseteq S''$ so that $U_{x,n} = U_{y,n}$ for any $x \in R, y \in T$.*

In the first step of the game, Empty selects an (i,j)-saturated pair R_0, T_0 below S' so that $U_{x,0} = U_{y,0}$ for any $x \in R_0, y \in T_0$. Let this common neighbourhood be U_0. By shrinking R_0, we can assume that all $x \in R_0$ is i-large in T_0. The set R_0 is the first choice of Empty by the strategy σ.

In general, given some large set $\sigma(2n-1)$, Empty picks an (i,j)-saturated pair R_σ, T_σ below $\sigma(2n-1)$ such that $U_{x,n} = U_{y,n}$ for any $x \in R_\sigma, y \in T_\sigma$. The common neighbourhood will be denote by U_n. Again, shrink R_σ so we can assume that all $x \in R_\sigma$ is i-large in T_σ. This R_σ is Empty's reply in the game.

Now, there must be a play so that Non Empty wins: there is a sequence (R_n, T_n) of (i,j)-saturated pairs below S' so that $\bigcap R_n \neq \emptyset$. Take any x from the intersection and let us show that x is an (i,j)-winner in S. Note that for any $n < \omega$, $S_n, T_n \subset U_{x,n} = U_n$ and $S_{n+1}, T_{n+1} \subset S_n$ and the set $T'_n = \{y \in T_n : c(x,y) = i\}$ is large. So, (R_n, T'_n) witnesses that x is an (i,j)-winner in S. This contradiction ends the proof.

□

□

We are ready to prove the theorem now: we construct a dense-in-itself subset $Y = \{x_{m+1} : m < \omega\}$ of X by induction. At each step $m < \omega$, we will have a subtree $P_m \subseteq \omega^{<\omega}$ of finite height along with large sets $T_{m,\sigma} \subseteq X$ attached to the leafs σ of P_m. We'll denote the set of leafs with $L(P_m)$. The sets $T_{m',\sigma}$ are reservoirs for points x_m that we choose in later stages.

We start from $P_0 = \{\emptyset\}$ and $T_{0,\emptyset} = X$. When forming P_{m+1} from P_m, we pick some leaf σ_m of P_m and let

$$P_{m+1} = P_m \cup \{\sigma_m{}^\frown(n) : n < \omega\}.$$

In the end, we would like that $\bigcup P_m = \omega^{<\omega}$ which is ensured by picking the leafs σ_m appropriately. I.e., for any $\sigma \in \omega^{<\omega}$, there will be some $m < \omega$ so that $\sigma = \sigma_m$.

We will also make sure that the following properties are satisfied by our construction:

(5) for any $m' \leq m$ and $\sigma \in L(P_{m'}) \cap L(P_m)$, $T_{m,\sigma} \subset T_{m',\sigma}$,
(6) for any $m < m'$ and $\sigma \in L(P_m)$,
 (a) if $\sigma_{m'} \subset \sigma$ then $c(x_{m'+1}, \cdot) \upharpoonright T_{m,\sigma} = i$,
 (b) if $\sigma_{m'} <_{lex} \sigma$ then $c(x_{m'+1}, \cdot) \upharpoonright T_{m,\sigma} = j$,
 (c) if $\sigma <_{lex} \sigma_{m'}$ then $c(x_{m'+1}, \cdot) \upharpoonright T_{m,\sigma} = i$,
(7) $\sigma <_{lex} \tau$ implies that $T_{m,\tau}, T_{m,\sigma}$ is an (i,j)-saturated pair,
(8) $x_{m+1} \in T_{m,\sigma_m}$,
(9) $T_{m+1,\sigma_m{}^\frown(n)} \subseteq T_{m,\sigma_m} \cap U_{x_{m+1},n}$.

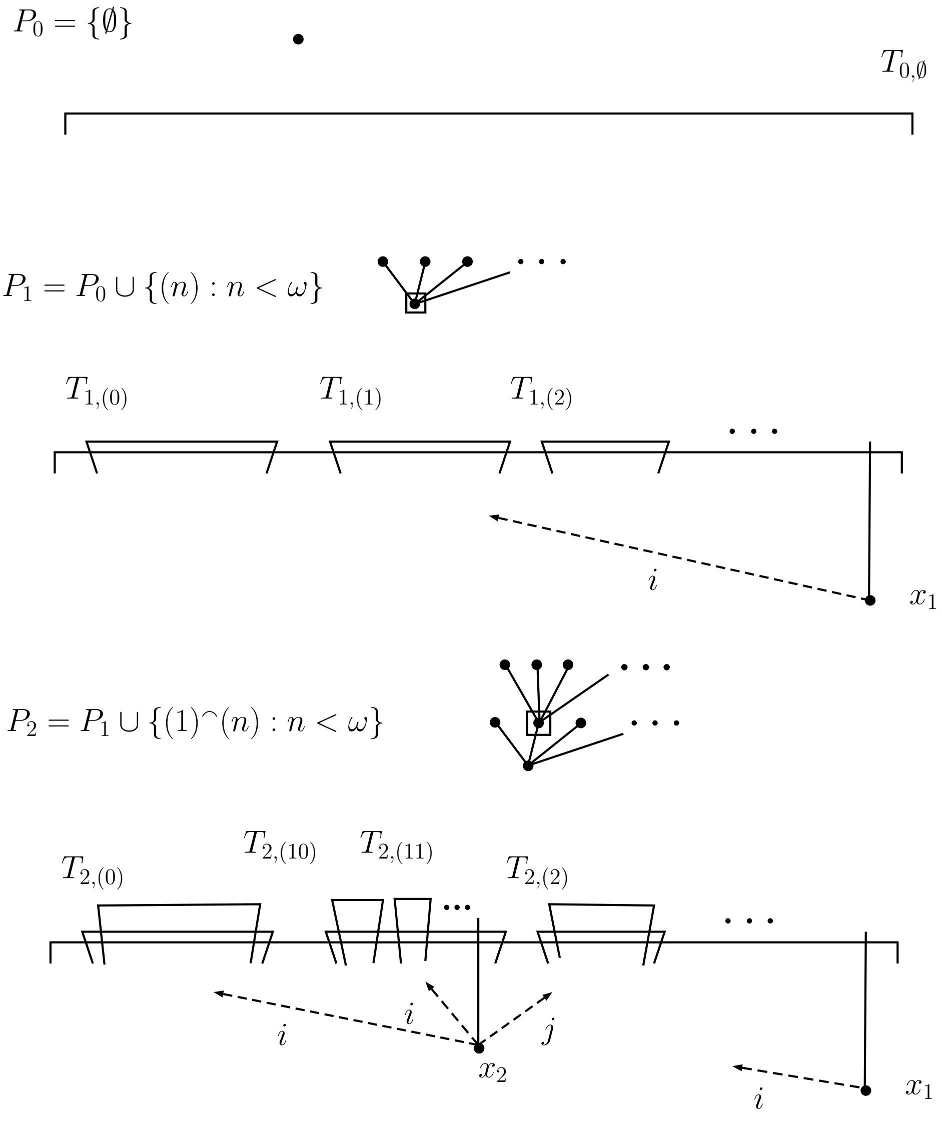

FIGURE 1. Construction the copy of $\mathbb{Q}$ with the auxiliary trees $P_0, P_1, \ldots$

We need to verify three things: that the construction can be carried out, the set Y is dense-in-itself and $\operatorname{ran} c \upharpoonright [Y]^2 \subseteq \{i, j\}$.

The construction is fairly straightforward and Figure 1 demonstrates the first few steps. In general, given P_m, we start by picking a leaf σ_m.

Lemma 0.9. *For almost every $x \in T_{m,\sigma_m}$ the following holds: for every $\sigma \in L(P_m) \setminus \{\sigma_m\}$, x is i-large in $T_{m,\sigma}$ if $\sigma <_{lex} \sigma_m$ and x is j-large in $T_{m,\sigma}$ if $\sigma_m <_{lex} \sigma$.*

Let T' denote the subset of T_{m,σ_m} where the above lemma holds. Now, apply Lemma 0.6 in T' to define $x_{m+1} \in T'$ and large sets $T_{m+1,\sigma_m{}^\frown(n)}$ for $n < \omega$ which converge to x_{m+1} and satisfy that $c(x_{m+1}, \cdot) \upharpoonright T_{m+1,\sigma_m{}^\frown(n)} = i$. Using that $x_{m+1} \in T'$, we can shrink $T_{m,\sigma}$ to large sets $T_{m+1,\sigma}$ for $\sigma \in L(P_m) \cap L(P_{m+1})$ so that condition (6) is satisfied.

Now, why is Y dense-in-itself? It suffices to note that after $x_{m'+1}$ was picked, for any $n < \omega$ there is $m' < m$ so that $\sigma_m = \sigma_{m'}{}^\frown(n)$ and so $x_m \in T_{m',\sigma'_m{}^\frown(n)}$. The sequence of sets $(T_{m',\sigma_{m'}{}^\frown(n)})_{n<\omega}$ converges to x_m, so we are done.

Finally, condition (6) and the fact that $m' < m$ implies $x_{m+1} \in \bigcup_{\sigma \in L(P_m)} T_{m,\sigma}$ ensure that $c(x_{m'+1}, x_{m+1}) \in \{i, j\}$.

References

[1] Matthew Foreman. Ideals and generic elementary embeddings. In *Handbook of set theory*, pages 885–1147. Springer, 2010.

[2] Thomas Jech, Menachem Magidor, William Mitchell, and Karel Prikry. Precipitous ideals. *The Journal of Symbolic Logic*, 45(1):1–8, 1980.

[3] Dilip Raghavan and Stevo Todorcevic. Proof of a conjecture of Galvin. *arXiv preprint arXiv:1809.00922*, 2018.

(D.T. Soukup) Universität Wien, Kurt Gödel Research Center for Mathematical Logic, Wien, Austria

E-mail address, Corresponding author: `daniel.soukup@univie.ac.at`

URL: `http://www.logic.univie.ac.at/~soukupd73/`