SUMS AND ANTI-RAMSEY COLOURINGS OF $\mathbb{R}$

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Abstract. Hindman, Leader and Strauss [1] proved that CH implies that there is a colouring $F : \mathbb{R} \to 2$ such that $F''\{x + y : x \neq y \in X\} = 2$ for any uncountable $X \subseteq \mathbb{R}$. Answering a problem from [1], we show that the same results holds in ZFC.

1. Introduction

In [1], the authors proved the following:

Theorem 1.1. *Assuming the Continuum Hypothesis there is a colouring* $F : \mathbb{R} \to 2$ *such that* F *is not monochromatic on any set* $\{\sum E : E \in [X]^N\} = 2$ *for any uncountable* $X \subseteq \mathbb{R}$ *and* $N \in \mathbb{N} \setminus \{0\}$.

The most natural question (also appearing in [1]) is if the assumption of CH can be omitted; we will show that the answer is yes. The same result was independently proved by P. Komjáth [2]. Also, we discuss if the number of colours can be increased to three or more.

2. Sums versus unions

First, let us prove that regarding this problem it is equivalent to work with addition in $\mathbb{R}$ or with unions in $[2^\omega]^{<\omega}$.

Lemma 2.1. *Let* ν *be a cardinal. Consider the following statements:*

(1) *there is a map* $f : [2^\omega]^{<\omega} \to \nu$ *such that for any uncountable* $X \subseteq [2^\omega]^{<\omega}$, $N \in \omega \setminus 2$ *and* $i < \nu$ *there are distinct* $a_0, ..., a_{N-1} \in X$ *with* $f(\bigcup_{j<N} a_j) = i$;

(2) *there is a colouring* $F : \mathbb{R} \to \nu$ *such that*

$$F''\{\sum E : E \in [X]^N\} = \nu$$

for any uncountable $X \subseteq \mathbb{R}$ *and* $N \in \omega \setminus 2$;

(3) $2^{\aleph_0} \not\to [\omega_1]^2_\nu$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3).

We remark that (1) $\Rightarrow$ (2) was essentially proved in [1].

Proof. Take a basis $\mathcal{B} = \{r_x : x \in 2^\omega\}$ of $\mathbb{R}$ over $\mathbb{Q}$.

(1) $\Rightarrow$ (2) : Let $f : [2^\omega]^{<\omega} \to 2$ witness (1). For any $r \in \mathbb{R}$, let $\mathrm{supp}(r)$ denote the finite set of $x \in 2^\omega$ such that r_x has a non zero coefficient in the expression of r as a linear combination of elements from $\mathcal{B}$. We define $F(r) = f(\mathrm{supp}(r))$ for any $r \in \mathbb{R}$. It suffices to note that for any uncountable $X \subseteq \mathbb{R}$, we can select a $Y \in [X]^{\omega_1}$ such that

(i) $r \to \mathrm{supp}(r)$ is 1-1 on Y, and
(ii) if $r_0, ..., r_{N-1} \in Y$ are distinct then $\mathrm{supp}(\sum_{i<N} r_i) = \bigcup_{i<n} \mathrm{supp}(r_i)$.

(2) $\Rightarrow$ (1) : let F witness (2) and define $f(a) = F(\sum_{x\in a} r_x)$ for any $a \in [\mathbb{R}]^{<\omega}$. Now, suppose that $X \subseteq [2^\omega]^{<\omega}$ is uncountable and fix $N \in \omega \setminus 2$ and $i < \nu$. Without loss of generality, we can suppose that X is a Δ-system with root d. Now, let

$$t_a = \frac{1}{N} \sum_{x\in d} r_x + \sum_{x \in a\setminus d} r_x$$

for $a \in X$. The set $\{t_a : a \in X\}$ is uncountable so there are $a_0, \dots, a_{N-1} \in X$ such that $F(\sum_{j<N} t_{a_j}) = i$. However, note that

$$\sum_{j<N} t_{a_j} = \sum_{x \in a_0 \cup \dots \cup a_{N-1}} r_x$$

and hence $f(\bigcup_{j<N} a_j) = F(\sum_{x\in a_0\cup\dots\cup a_{N-1}} r_x) = i$.

(1) $\Rightarrow$ (3) : let f witness (1) and simply define F as the restriction of f to $[\mathbb{R}]^2$.

□

This lemma has two immediate corollaries:

Corollary 2.2. *CH implies that there is a colouring $F : \mathbb{R} \to 2^\omega$ such that*

$$F''\{\sum E : E \in [X]^N\} = 2^\omega$$

for any uncountable $X \subseteq \mathbb{R}$ and $N \in \omega \setminus 2$.

Proof. Lemma 5.2.6 [4] implies that there is a map $c : [\omega_1]^{<\omega} \to \omega_1$ such that for any uncountable $X \subseteq [\omega_1]^{<\omega}$ and $i < \omega_1$ there are $a \neq b \in X$ with $c(a \cup b) = i$. Hence (1) and (2) from Lemma 2.1 holds with $\nu = \omega_1$ if the Continuum Hypothesis is true. □

This improvement of Theorem 1.1 is clearly the best possible. Also, a simple forcing argument shows that the same result can hold with a large continuum.

On the other hand:

Corollary 2.3. *Consistently, the number of colours in Theorem 1.1 cannot be increased to three.*

Proof. Shelah [3] proved that consistently $2^{\aleph_0} \to [\omega_1]^2_3$ (with $2^{\aleph_0} = \aleph_2$) hence consistently (1) and (2) of Lemma 2.1 might fail with three colours. □

3. A 2-colouring in ZFC

Now, we show that (1) from Lemma 2.1 holds with $\nu = 2$ in ZFC.

Theorem 3.1. *There is a map $f : [2^\omega]^{<\omega} \to 2$ such that for any uncountable $X \subseteq [2^\omega]^{<\omega}$, $N \in \omega \setminus 2$ and $i < 2$ there are distinct $a_0, ..., a_{N-1} \in X$ with $f(\bigcup_{j<N} a_j) = i$.*

We use an idea from the proof of Lemma 5.2.6 [4].

Proof. Let $\Delta(x, y) = \min\{n : x(n) \neq y(n)\}$ for incomparable $x, y \in 2^{\leq\omega}$ and $\Delta(a) = \max\{\Delta(x, y) : x \neq y \in a\}$ for any finite $a \subseteq 2^{\leq\omega}$ with at least two (incomparable) elements. Let $\pi(a)$ be the $<_\mathbb{R}$-minimal $\{x, y\} \in [a]^2$ such that $\Delta(a) = \Delta(x, y)$. Let $g : [2^\omega]^2 \to 2$ denote the Sierpinski colouring i.e. g compares the Euclidean ordering $<_\mathbb{R}$ and a well ordering $<_\mathfrak{c}$ on 2^ω. Define $f : [2^\omega]^{<\omega} \to 2$ as follows: let $f(a) = g(\pi(a))$ for $a \in [2^\omega]^{<\omega}, |a| \geq 2$ and $f(a) = 0$ otherwise.

Fix an uncountable $X \subseteq [2^\omega]^{<\omega}$. Without loss of generality, we can write X as $\{a_\xi : \xi < \omega_1\}$ such that there is $n \in \omega$ and pairwise incomparable $\varepsilon_k \in 2^{<\omega}$ for $k < n$ such that

(1) $a_\xi = \{a^k_\xi : k < n\}$ corresponds to the $<_\mathbb{R}$ increasing enumeration,
(2) $a_\xi \cap [\varepsilon_k] = \{a^k_\xi\}$ for each $k < n, \xi < \omega_1$, and
(3) $X_k = \{a^k_\xi : \xi < \omega_1\}$ is either a singleton or a strictly $<_\mathfrak{c}$ increasing sequence.

Note that $\Delta(a_\xi) = \Delta(\{\varepsilon_k : k < n\})$ for any $\xi < \omega_1$. Let $M \subseteq n$ denote the (non empty) set of $k < n$ such that X_k is not a singleton. Let $M = \{k_j : j < l\}$.

Let us prove the $N = 2$ case first. By induction on $j < l$, we define decreasing sequences $(\Gamma^0_j)_{j<l}$ and $(\Gamma^1_j)_{j<l}$ of uncountable subsets of ω_1 such that there are incomparable extensions $\varepsilon^0_{k_j}, \varepsilon^1_{k_j}$ of ε_{k_j} with the property that $\xi \in \Gamma^i_j$ implies $a^{k_j}_\xi \in [\varepsilon^i_{k_j}]$. Let $\Gamma^i = \Gamma^i_{k_{l-1}}$ for $i < 2$ and we can suppose that $\Gamma^0 \cap \Gamma^1 = \emptyset$. Let $\varepsilon^0_k = \varepsilon^1_k = \varepsilon_k$ for $k \in n \setminus M$.

Now, let

$$m = \Delta(\{\varepsilon^0_k, \varepsilon^1_k : k < n\})$$

and note that

$$m = \Delta(a_\xi \cup a_\zeta) > \Delta(a_\xi) = \Delta(a_\zeta)$$

for all $\xi \in \Gamma^0$ and $\zeta \in \Gamma^1$. There is a minimal $k^* \in M$ such that $m = \Delta(\varepsilon^0_{k^*}, \varepsilon^1_{k^*})$; indeed, if $k \neq l < n$ then $\Delta(\varepsilon_k, \varepsilon_l) < m$ as noted above. In turn, if $\xi \in \Gamma^0$ and $\zeta \in \Gamma^1$ then

$$\pi(a_\xi \cup a_\zeta) = \{a^{k^*}_\xi, a^{k^*}_\zeta\}.$$

As Γ^0 and Γ^1 are both uncountable, we can find $\xi \in \Gamma^0$ and $\zeta \in \Gamma^1$ such that $a^{k^*}_\xi <_{\mathfrak{c}} a^{k^*}_\zeta$ (i.e. choose $\xi < \zeta$) and $\mu \in \Gamma^0$ and $\nu \in \Gamma^1$ such that $a^{k^*}_\nu <_{\mathfrak{c}} a^{k^*}_\mu$ (choose $\mu > \nu$).

Note that

$$g(\{a^{k^*}_\nu, a^{k^*}_\mu\}) = 1 - g(\{a^{k^*}_\xi, a^{k^*}_\zeta\})$$

and hence

$$f(a_\mu \cup a_\nu) = 1 - f(a_\xi \cup a_\zeta).$$

This finishes the proof for $N = 2$.

In general for $N \geq 2$, instead of selecting Γ^0 and Γ^1 we find pairwise disjoint uncountable $\Gamma^0, \dots, \Gamma^{N-1} \subseteq \omega_1$ and incomparable extensions $\varepsilon^0_{k_j}, \dots, \varepsilon^{N-1}_{k_j}$ of ε_{k_j} with the property that $\xi \in \Gamma^i$ implies $a^{k_j}_\xi \in [\varepsilon^i_{k_j}]$ for all $j \in M$ and $i < N$. We define

$$m = \Delta(\{\varepsilon^i_k : k < n, i < N\})$$

and note that

$$\Delta(\bigcup_{i<N} a_{\xi_i}) = m$$

for all $(\xi_i)_{i<N} \in \Pi_{i<N}\Gamma_i$.

Now, find minimal $k^* \in M$ and $i_0, i_1 < N$ such that $m = \Delta(\varepsilon^{i_0}_{k^*}, \varepsilon^{i_1}_{k^*})$. Observe that if $(\xi_i)_{i<N} \in \Pi_{i<N}\Gamma_i$ then

$$\pi(\bigcup_{i<N} a_{\xi_i}) = \{a^{k^*}_{\xi_{i_0}}, a^{k^*}_{\xi_{i_1}}\}.$$

To finish the proof, it suffices to repeat the last paragraph of the $N = 2$ case with Γ^{i_0} and Γ^{i_1}. □

Corollary 3.2. *There is a colouring $F : \mathbb{R} \to 2$ such that*

$$F''\{\sum E : E \in [X]^N\} = 2$$

for any uncountable $X \subseteq \mathbb{R}$ and $N \in \omega \setminus 2$.

4. Open problems

Let us mention another result of similar flavour from [1]:

Theorem 4.1. *If $2^\omega < \aleph_\omega$ then there is a finite colouring f of $\mathbb{R}$ such that f is not constant on any set of the form $X + X$ where $X \subseteq \mathbb{R}$ is infinite.*

It is not known if the cardinal arithmetic assumption can be removed from this result.

References

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