Sums and anti-Ramsey colourings

Dániel T. Soukup

http://renyi.hu/∼dsoukup/

Introduction

Goal: find anti-Ramsey colourings of $(\mathbb{R}, +)$.

- the origins of the problem
- failure of the classical Ramsey-theorem on $\mathbb{R}$,
- recent results and open problems.

Joint work with W. Weiss and Z. Vidnyánszky.

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The original problem

Problem [J.C. Owings]: Is there a colouring $f : \mathbb{N} \to 2$ such that f **is not constant on** $X + X$ whenever X is infinite?

- **there is a 3-colouring** with such properties,
- there are always **arbitrary large finite** sets X such that f is constant on $X + X$,
- **[Hindman]** For every colouring $f : \mathbb{N} \to r$ there is an infinite $X \subseteq \mathbb{N}$ such that f **is constant** on

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A 4-colouring for the Owings problem

Let $f : \mathbb{N} \to 4$ defined as

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \pmod 4.$$

Proof:

- let $X \subseteq \mathbb{N}$ be infinite,
- find $x << y \in X$ such that $|\log_{\sqrt{2}}(y) - \log_{\sqrt{2}}(y+x)| < 1$,
- $\Rightarrow |\lfloor \log_{\sqrt{2}}(y) - \log_{\sqrt{2}}(y+x) \rfloor| \leq 1$, so $f(y+x) = f(y) \pm 1$,
- but note that $f(2y) = f(y) + 2$ modulo 4 so $f(2y) \neq f(x+y)$.

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[Sierpinski, 1933] There is a colouring $f : [\mathbb{R}]^2 \to 2$ such that f is **not constant on** $[X]^2$ whenever $X \subseteq \mathbb{R}$ is **uncountable**.

Proof:

- let $\prec$ denote a linear ordering of $\mathbb{R}$ **without infinite decreasing chains** and maximal element,
- define $f(x, y) = 0$ **iff** $x <_{\mathbb{R}} y$ **and** $x \prec y$ both holds,
- let X be uncountable and suppose that $f''[X]^2 = 0$,
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Additive anti-Ramsey results on $\mathbb{R}$

[Hindman, Leader, Strauss 2015] Analyze the Owings problem for **colourings of** $\mathbb{R}$ with finitely many colours.

Results:

- **[D.S., W. Weiss]** There is a colouring $f : \mathbb{R} \to 2$ such that $f \restriction \{x + y : x \neq y \in X\}$ **is not constant** for any uncountable $X \subseteq \mathbb{R}$.
  - **[Hindman, Leader, Strauss]** proved it using the Continuum Hypothesis, we removed this assumption,
  - we can't realize 3 colours as well (even more set theory comes in).
- **[Hindman, Leader, Strauss]** The Continuum Hypothesis implies that there is a colouring $f : \mathbb{R} \to 288$ such that $f \restriction X + X$ **is not constant** for any infinite $X \subseteq \mathbb{R}$.

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How is the Continuum Hypothesis used?

Recall: the **Continuum Hypothesis** says that $|\mathbb{R}|$ is the smallest uncountable cardinality.

Note that $\mathbb{R} = \bigoplus_{\mathfrak{c}} \mathbb{Q}$.

What can we say about $\mathbb{Q}$ itself or finite/countable direct sums?

From **[Hindman, Leader, Strauss]**:

- For any $m \in \mathbb{N}$, there is $f : \bigoplus_m \mathbb{Q} \to 72$ such that $f \restriction X + X$ is not constant for any infinite $X \subseteq \bigoplus_m \mathbb{Q}$;
- Step up lemma: if $N \in \mathbb{N}$ fixed and $\bigoplus_\kappa \mathbb{Q}$ has a good N-colouring for every $\kappa < \lambda$ then $\bigoplus_\lambda \mathbb{Q}$ has a good $2N$-colouring.
- Corollary: we can find good colourings for the first countably many cardinalities before the number of colours blows up...

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Recall: the **Continuum Hypothesis** says that $|\mathbb{R}|$ is the smallest uncountable cardinality.

Note that $\mathbb{R} = \bigoplus_{\mathfrak{c}} \mathbb{Q}$.

What can we say about $\mathbb{Q}$ itself or finite/countable direct sums?

From **[Hindman, Leader, Strauss]**:

- For any $m \in \mathbb{N}$, there is $f : \bigoplus_m \mathbb{Q} \to 72$ such that $f \restriction X + X$ is not constant for any infinite $X \subseteq \bigoplus_m \mathbb{Q}$;
- Step up lemma: if $N \in \mathbb{N}$ fixed and $\bigoplus_\kappa \mathbb{Q}$ has a good N-colouring for every $\kappa < \lambda$ then $\bigoplus_\lambda \mathbb{Q}$ has a good $2N$-colouring.
- Corollary: we can find good colourings for the first countably many cardinalities before the number of colours blows up...

How to proceed?

Remove CH and/or find $f : \bigoplus_\kappa \mathbb{Q} \to N$ for any κ!?

Let I be an ordered set.

- We say that $x \sim y \in \bigoplus_I \mathbb{Q}$ iff there is an order isomorphism $\varphi : \mathrm{supp}(x) \to \mathrm{supp}(y)$ such that $x(i) = y(\varphi(i))$;
- A colouring f of $\bigoplus_I \mathbb{Q}$ is **strong** iff
  - $f(x) = f(y)$ whenever $x \sim y$,
  - $f \restriction X + X$ is not constant for any infinite $X \subseteq \bigoplus_I \mathbb{Q}$.

Lemma

If there is a strong colouring $f : \bigoplus_{\mathbb{Q}} \mathbb{Q} \to N$ (for some finite N) then there is a strong colouring $\bigoplus_\kappa \mathbb{Q} \to N$ for any κ.

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Open problems

Recall: f is strong iff $f(x) = f(y)$ whenever $x \sim y$ and $f \restriction X + X \neq$ constant for any infinite X.

Open Problem

Is there a strong colouring $f : \bigoplus_{\mathbb{Q}} \mathbb{Q} \to N$ for some finite N?

Recall: YES $\Rightarrow$ every $\bigoplus_{\kappa} \mathbb{Q}$ has a strong colouring.

Open Problem

Is there a good colouring of $f : \mathbb{R} \to N$ for some finite N without CH?

Is there a smarter step-up lemma maybe?

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Thank you for your attention.

Any questions?