

Sums and anti-Ramsey colourings

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Goal: find **anti-Ramsey colourings** of $(\mathbb{R}, +)$.

- **the origins** of the problem
- failure of the classical Ramsey-theorem on \mathbb{R} ,
- **recent results and open problems.**

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The original problem

Problem [J.C. Owings]: Is there a colouring $f : \mathbb{N} \rightarrow 2$ such that f is not constant on $X + X$ whenever X is infinite?

- there is a 3-colouring with such properties,
- there are always arbitrary large finite sets X such that f is constant on $X + X$,
- [Hindman] For every colouring $f : \mathbb{N} \rightarrow r$ there is an infinite $X \subseteq \mathbb{N}$ such that f is constant on

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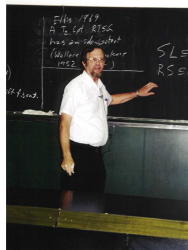
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A 4-colouring for the Owings problem

Let $f : \mathbb{N} \rightarrow 4$ defined as

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \pmod{4}.$$

Proof:

- let $X \subseteq \mathbb{N}$ be infinite,
- find $x \ll y \in X$ such that $|\log_{\sqrt{2}}(y) - \log_{\sqrt{2}}(y+x)| < 1$,
- $\Rightarrow \lfloor \log_{\sqrt{2}}(y) - \log_{\sqrt{2}}(y+x) \rfloor \leq 1$, so $f(y+x) = f(y) \pm 1$,
- but note that $f(2y) = f(y) + 2$ modulo 4 so $f(2y) \neq f(x+y)$.

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Classical anti-Ramsey on \mathbb{R}

[Sierpinski, 1933] There is a colouring $f : [\mathbb{R}]^2 \rightarrow 2$ such that f is **not constant on $[X]^2$** whenever $X \subseteq \mathbb{R}$ is **uncountable**.

Proof:

- let \prec denote a linear ordering of \mathbb{R} **without infinite decreasing chains** and maximal element,
- define $f(x, y) = 0$ iff $x <_{\mathbb{R}} y$ and $x \prec y$ both holds,
- let X be uncountable and suppose that $f''[X]^2 = 0$,
- let x^+ denote the immediate \prec -successor of x ,
- pick a $q_x \in \mathbb{Q} \cap (x, x^+)$ (note that $x <_{\mathbb{R}} x^+$ as $f(x, x^+) = 0$),
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Additive anti-Ramsey results on \mathbb{R}

[Hindman, Leader, Strauss 2015] Analyze the Owings problem for colourings of \mathbb{R} with finitely many colours.

Results:

- [D.S., W. Weiss] There is a colouring $f : \mathbb{R} \rightarrow 2$ such that $f \upharpoonright \{x + y : x \neq y \in X\}$ is not constant for any uncountable $X \subseteq \mathbb{R}$.
 - [Hindman, Leader, Strauss] proved it using the Continuum Hypothesis, we removed this assumption.
 - we can't realize 3 colours as well (even more set theory comes in).
- [Hindman, Leader, Strauss] The Continuum Hypothesis implies that there is a colouring $f : \mathbb{R} \rightarrow 288$ such that $f \upharpoonright X + X$ is not constant for any infinite $X \subseteq \mathbb{R}$.

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How is the Continuum Hypothesis used?

Recall: the **Continuum Hypothesis** says that $|\mathbb{R}|$ is the smallest uncountable cardinality.

Note that $\mathbb{R} = \bigoplus_c \mathbb{Q}$.

What can we say about \mathbb{Q} itself or finite/countable direct sums?

From [**Hindman, Leader, Strauss**]:

- For any $m \in \mathbb{N}$, there is $f : \bigoplus_m \mathbb{Q} \rightarrow 72$ such that $f \upharpoonright X + X$ is not constant for any infinite $X \subseteq \bigoplus_m \mathbb{Q}$;
- Step up lemma: if $N \in \mathbb{N}$ fixed and $\bigoplus_\kappa \mathbb{Q}$ has a good N -colouring for every $\kappa < \lambda$ then $\bigoplus_\lambda \mathbb{Q}$ has a good $2N$ -colouring.
- Corollary: we can find good colourings for the first countably many cardinalities before the number of colours blows up...

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- Step up lemma: if $N \in \mathbb{N}$ fixed and $\bigoplus_\kappa \mathbb{Q}$ has a good N -colouring for every $\kappa < \lambda$ then $\bigoplus_\lambda \mathbb{Q}$ has a good $2N$ -colouring.
- Corollary: we can find good colourings for the first countably many cardinalities before the number of colours blows up...

How to proceed?

Remove CH and/or find $f : \bigoplus_{\kappa} \mathbb{Q} \rightarrow N$ for any κ !?

Let I be an ordered set.

- We say that $x \sim y \in \bigoplus_I \mathbb{Q}$ iff there is an order isomorphism $\varphi : \text{supp}(x) \rightarrow \text{supp}(y)$ such that $x(i) = y(\varphi(i))$;
- A colouring f of $\bigoplus_I \mathbb{Q}$ is **strong** iff
 - $f(x) = f(y)$ whenever $x \sim y$,
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Lemma

If **there is a strong colouring** $f : \bigoplus_{\aleph_0} \mathbb{Q} \rightarrow N$ (for some finite N) then there is a strong colouring $\bigoplus_{\kappa} \mathbb{Q} \rightarrow N$ for any κ .

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Open problems

Recall: f is strong iff $f(x) = f(y)$ whenever $x \sim y$ and $f \upharpoonright X + X \neq \text{constant}$ for any infinite X .

Open Problem

Is there a strong colouring $f : \bigoplus_{\mathbb{Q}} \mathbb{Q} \rightarrow N$ for some finite N ?

Recall: YES \Rightarrow every $\bigoplus_{\kappa} \mathbb{Q}$ has a strong colouring.

Open Problem

Is there a good colouring of $f : \mathbb{R} \rightarrow N$ for some finite N without CH?

Is there a smarter step-up lemma maybe?

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Thank you for your attention.

Any questions?