Colouring problems of Erdős and Rado on infinite graphs

by

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Abstract

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The aim of this thesis is to provide solutions to two old problems on infinite graphs. First, we investigate vertex partitions of edge coloured complete graphs into monochromatic paths. Our main result here is an answer to a question of R. Rado from 1978: we prove that every finite edge coloured infinite complete graph can be partitioned into disjoint monochromatic paths of different colours. We also show that analogous results hold for partitions into powers of paths and for hypergraphs in the countably infinite case. Second, we turn to the theory of uncountably chromatic graphs and obligatory subgraphs. We mainly focus on the question whether every graph with large chromatic number contains highly connected subgraphs. With the aid of a special sequence of countable elementary submodels (called Davies-trees or ω_1-approximation sequences), we present a new and highly simplified proof of Komjáth's theorem: every graph with uncountable chromatic number contains an n-connected subgraph with uncountable chromatic number (for each $n \in \mathbb{N}$). We outline multiple new applications of Davies-trees to combinatorial problems. Finally, our most important contribution to the theory is an answer to a popular problem of A. Hajnal and P. Erdős from 1985: we construct a graph of chromatic number ω_1 without an uncountable infinitely connected subgraph. A general and rather flexible method is introduced which uses ladder systems on non special trees in ZFC. This machinery is applied again to present an example of a triangle-and $H_{\omega,\omega+2}$-free graph with uncountable chromatic number.

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Chapter 1

Introduction

In Chapter 1, we present a detailed outline of the thesis followed by an introduction to our notations and certain preliminary results.

1.1 An overview of the main results

The goal of this thesis is to investigate two topics on infinite graphs: path decompositions of edge coloured complete graphs and the relations between having large chromatic number and connectivity.

The topic of path decompositions grew out of a paper of R. Rado [78] where he proved that for every finite-edge colouring of the complete graph on $\mathbb{N}$ (denoted by $K_{\mathbb{N}}$) the vertices can be partitioned into disjoint monochromatic paths of different colours. In Chapter 2 and 3, we generalize Rado's result in multiple ways both on countable and uncountable vertex sets. The work in Chapter 2 was jointly done with M. Elekes, L. Soukup and Z. Szentmiklóssy and the results are currently submitted to Combinatorica [14].

After an overview in Section 2.1 of the most important results preceding our work, we start by investigating monochromatic path decompositions of edge-coloured complete uniform hypergraphs in Section 2.2. Answering a question from [40], we prove

Theorem 2.2.2. *Suppose that the edges of a countably infinite complete k-uniform hypergraph are coloured with r colours. Then*

(1) the vertex set can be partitioned into monochromatic finite or one-way infinite tight paths of distinct colours,

(2) the vertex set can be partitioned into monochromatic tight cycles and two-way infinite tight paths of distinct colours.

Erdős, Gyárfás and Pyber [19] conjectured that the vertices of every r-edge coloured finite complete graph can be covered with r disjoint monochromatic cycles. This was recently disproved by Pokrovskiy [77], however, the case $k = 2$ of Theorem 2.2.2(2) gives a positive answer for the infinite case.

Second, we turn to decompositions of edge-coloured copies of $K_{\mathbb{N}}$ into monochromatic k^{th} powers of paths. Our main result in Section 2.3 is

Theorem 2.3.6. *For all positive natural numbers k, r and an r-edge colouring of $K_{\mathbb{N}}$ the vertices can be covered by $\leq r^{(k-1)r+1}$ one-way infinite disjoint monochromatic k^{th} powers of paths and a finite set.*

In the case of $k = r = 2$, we have the following stronger result:

Theorem 2.3.7. *Given an edge colouring of $K_{\mathbb{N}}$ with 2 colours, the vertices can be partitioned into $\leq$ 4 monochromatic path-squares (that is, second powers of paths). Moreover, there is an edge colouring of $K_{\mathbb{N}}$ with 2 colours such that the vertices cannot be covered by 3 monochromatic path-squares.*

We introduce a two-player game $\mathfrak{G}_k(H, W)$ on a graph H with parameters $W \subseteq V(H)$ and $k \in \mathbb{N}$. A winning strategy for Player II (Bob, in our case) will yield a cover of W by a k^{th}-power of a path. We first find conditions on a set W that ensure that Player II has a winning strategy in the game $\mathfrak{G}_k(H, W)$ (see Theorem 2.3.5) and then apply this result in the proofs of Theorem 2.3.6 and Theorem 2.3.7.

Finally, in Section 2.4, we mention some open problems and say a few words about non-complete graphs: we prove a conjecture of A. Pokrovskiy [77] for the countably infinite balanced bipartite graph.

Chapter 3 is devoted to answering a question of R. Rado [78]. We show

Theorem 3.4.1. *Suppose that c is a finite-edge colouring of an infinite graph $G = (V, E)$ which satisfies*

$$|\{v \in V : \{u, v\} \notin E\}| < |V|$$

for all $u \in V$. Then the vertices of G can be partitioned into disjoint monochromatic paths of different colours.

In particular, Rado's theorem extends to arbitrary infinite complete graphs. This theorem is proved through a series of lemmas and theorems on finding monochromatic paths in certain classes of graphs. We emphasize Lemma 3.1.8 from Section 3.1, where we show that any set of vertices A in a graph G which satisfies three rather simple properties can be covered by a path. Next, we prove Lemma 3.2.8 and 3.2.13 which imply the existence of large sets satisfying all three conditions of Lemma 3.1.8; this is achieved in Theorem 3.2.20 in Section 3.2. After further preparations in Section 3.3, the previous results are finally put together in Section 3.4 in the proof of Theorem 3.4.1.

In Chapter 4 we turn to chromatic number problems. In general, we are interested in the question if a graph with large chromatic number must have certain obligatory subgraphs. After presenting a short history of this problem in Section 4.1, we outline the most fundamental results about obligatory subgraphs of graphs with uncountable chromatic number in Section 4.2. Here, we present highly simplified proofs to several classical results using elementary submodels.

In Section 4.3, we generalize a result of P. Erdős and A. Hajnal [20] on paths and chromatic number by proving the following results:

Corollary 4.3.4. *Every graph G with $Col(G) > \omega$ contains a path of order type ξ for all $\xi < \omega_1$.*

Theorem 4.3.10. *Suppose that MA_κ holds. Then every graph G with $Chr(G) > \omega$ and size $< \kappa$ contains a path of size ω_1.*

However, the above cited theorem cannot be extended to graphs of size 2^ω:

Corollary 4.3.8. *There is a graph G of size 2^ω and chromatic number ω_1 such that every path in G is countable.*

In Chapter 5, we turn to the question if graphs with uncountable chromatic number must contain large highly connected subgraphs. This problem first appeared in [20] and received significant attention in the past [55, 57, 62, 22, 63, 61]. The main result of Chapter 5 is a new proof of the following theorem of Komjáth presented in Section 5.3:

Theorem 5.3.1 ([55]). *Every uncountably chromatic graph G contains n-connected uncountably chromatic subgraphs for every $n \in \mathbb{N}$.*

We use a special sequence of countable elementary submodels, called Davies-trees, in order to present a highly simplified argument. We introduce the reader to Davies-trees in Section 5.2 and overview previous applications. We finish Chapter 5 in Section 5.4 with further new applications of Davies-trees to combinatorial problems.

In Chapter 6, the last chapter of the thesis, we answer an old problem of Erdős and Hajnal [22] by proving:

Theorem 6.3.5. *There is a graph X of size 2^ω and chromatic number ω_1 such that X contains no uncountable infinitely connected subsets.*

The history of this problem is summarized in Section 5.1 and the above theorem is proved in Section 6.3 after preparations in Section 6.2. We will prove Theorem 6.3.5 by considering comparability graphs of certain non special trees T. That is, we look at the graph $G(T) = (T, E)$ where $\{u, v\} \in E$ iff $u \leq_T v$. We show that there is a ladder system on our tree T which gives a subgraph X of $G(T)$ with the required properties. While this particular technique seems to be new, similar methods appeared in many places; we overview the literature in Section 6.1.

The techniques we apply in Chapter 6 turn out to be rather flexible. First, in Section 6.4 we strengthen Theorem 6.3.5:

Theorem 6.4.3. *There is a graph X of size 2^ω and chromatic number ω_1 such that every uncountable set of vertices A in X contains two vertices which are separated by a finite set in X.*

Again, X is a carefully constructed subgraph of $G(T)$ (where T is a non special tree) and now we make sure that any two $<_T$-incomparable points are separated by a finite set.

Second, we present a triangle free graph:

Theorem 6.5.5. *There is a graph X of size 2^ω and chromatic number ω_1 such that X contains no triangles or copies of $H_{\omega,\omega+2}$.*

As before, X is a subgraph of a comparability graph $G(T)$ induced by a ladder system on T. In Theorem 6.5.5, we make sure that the graph X contains no cycles which are the union of two $\leq_T$-monotone paths which in turn ensures that X is triangle-and $H_{\omega,\omega+2}$-free. We use the framework developed by Hajnal and Komjáth [42].

Finally, in Section 6.6 and 6.7 we close with a few general remarks and problems about ladder subgraphs of comparability graphs.

1.2 Notations

A *graph* is an ordered pair $G = (V, E)$ so that $E \subseteq [V]^2$; we will use the notation $V(G), E(G)$ for the vertices and edges of a graph G. A *hypergraph* is an ordered pair $H = (V, E)$ so that $E \subseteq \mathcal{P}(V)$. We say that a hypergraph $H = (V, E)$ is *k-uniform* iff $E \subseteq [V]^k$.

For a graph $G = (V, E)$ we write

$$N_G(v) = \{w \in V : \{v, w\} \in E\}$$

for $v \in V$ and

$$N_G[F] = \bigcap\{N_G(v) : v \in F\}$$

for $F \subseteq V$.

We say that H is a *subgraph* of G iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Whenever we say that "a graph G contains a graph H" we mean that H is a subgraph of G. Given $W \subseteq V(G)$ we write $G \restriction W$ for the induced subgraph on W in G i.e. $G \restriction W = (W, \mathcal{P}(W) \cap E(G))$.

A set of vertices W in a graph G is said to be *independent* iff there is no edge $e \in E(G)$ with $e \subseteq W$. A set of edges $F \subseteq E(G)$ in a graph G is said to be *independent* iff F is a pairwise disjoint family as a subset of $[V]^2$.

An *r-edge colouring* of a graph $G = (V, E)$ is a map $c : E \to r$ where r is some cardinal. We write $c(v, w)$ instead of $c(\{v, w\})$ for an edge $\{v, w\} \in E$ for obvious reasons. A *finite edge colouring* is an r-edge colouring for some $r \in \mathbb{N}$. We will use the following notation: if we have a fixed edge colouring c of a graph $G = (V, E)$ then

$$N_G(v, i) = \{w \in N_G(v) : c(v, w) = i\}$$

for $v \in V$ and

$$N_G[F, i] = \bigcap\{N_G(v, i) : v \in F\}$$

for $F \subseteq V$ and $i \in \operatorname{ran} c$. As we always work with a single colouring one at a time, this notation will lead to no misunderstanding. If we work with a single graph then occasionally we omit the subscript G as well.

Let us fix an edge colouring c of G with r colours and $i < r$. If $\mathcal{P}$ is a graph property (e.g. being a path, being connected...) and $A \subseteq V$ then we say that

A has property $\mathcal{P}$ in colour i

with respect to c iff A has property $\mathcal{P}$ in the graph $(V, c^{-1}(i))$. In particular, by a *monochromatic path* in G we mean a subgraph P of $(V, c^{-1}(i))$ which is a path (for some $i < r$).

Let κ, λ be *ordinals*. Let $K_{\kappa,\lambda}$ denote the complete bipartite graph on classes of size κ and λ. We let $H_{\kappa,\lambda}$ denote the graph $(\kappa \times \{0\} \cup \lambda \times \{1\}, E)$ where

$$\{(\alpha, i), (\beta, j)\} \in E \iff i = 0, j = 1 \text{ and } \alpha < \kappa, \beta < \lambda \setminus \alpha.$$

$H_{\kappa,\lambda}$ is a bipartite graph and we call the set of vertices $\kappa \times \{0\}$ in $H_{\kappa,\lambda}$ *the main class of* $H_{\kappa,\lambda}$. If H denotes a copy of $H_{\kappa,\kappa}$ then let $H \restriction \alpha$ stand for $H_{\kappa,\kappa} \restriction \alpha \times 2$ for any $\alpha < \kappa$.

Throughout the thesis, we use standard set theoretic notations consistent with the literature, e.g. [67].

1.3 Infinite paths

A *path* in a graph G is a 1-1 sequence of vertices $v_0, v_1, \dots$ such that $\{v_i, v_{i+1}\} \in E(G)$. Let us recall how R. Rado defined paths of arbitrary length.

Definition 1.3.1 (R. Rado, [78]). *We say that a graph P is a* path *iff there is a well ordering $<_P$ on $V(P)$ such that*

$$\{w \in N_P(v) : w <_P v\} \text{ is } <_P\text{-cofinal below } v$$

for all $v \in V(P)$.

Observation 1.3.2. *A graph P is a path witnessed by the well ordering $<_P$ iff for all $v <_P w \in V(P)$ there is a $<_P$-monotone finite path from v to w.*

In particular, two vertices are connected by a transfinite path if and only if they are connected by a finite path.

We call the order type of $(V(P), <_P)$ above *the order type of P*. If P is a path of order type κ then we let $P \restriction \alpha$ denote the unique initial segment of P of order type α (for any $\alpha < \kappa$). Similarly, if $q \in P$ then let $P \restriction q = P \restriction \{p \in V(P) : p <_P q\}$.

We will say that a path Q *end extends* the path P iff $P \subseteq Q$, $<_Q \restriction V(P) = <_P$ and $v <_Q w$ for all $v \in V(P), w \in V(Q) \setminus V(P)$.

If R, S are two paths so that the first point of S has $<_R$-cofinally many neighbours in R then $R \cup S$ is a path which end extends R and we denote this path by $R^\frown S$ emphasizing this relation.

1.3.1 Paths and connectivity

It is not surprising that notions of *connectivity* are closely related to paths. Let us introduce some terminology:

Definition 1.3.3. *Let $G = (V, E)$ be a graph, κ a cardinal and let $A \subseteq V$. We say that A is* κ-unseparable *iff there are κ-many pairwise disjoint finite paths in G between any two points of A. We say that A is* κ-connected *iff there are κ-many pairwise disjoint finite paths in $G \restriction A$ between any two points of A.*

The following is obvious:

Observation 1.3.4. *Every ω-connected countable graph is a path of order type ω. Every countable ω-unseparable set is covered by a path of order type $\leq \omega$.*

The next lemma describes a method to find connected subsets of edge coloured graphs and was essentially proved in [44].

Lemma 1.3.5. *Suppose that $G = (V, E)$ is a graph, $A \in [V]^\omega$ and $N_G[F]$ is infinite for all $F \in [A]^{<\omega}$. Given any edge colouring $c : E \to r$ with $r \in \omega$, there is a partition $d_c : V \to r$ and a colour $i_c < r$ so that*

$$N[F, i] \cap V_{i_c} \text{ is infinite for all } i < r \text{ and finite set } F \subset A \cap V_i \text{ where } V_i = d_c^{-1}\{i\}.$$

In particular, $A \cap V_i$ is ω-unseparable in colour i for all $i < r$ and if $V = A$ then V_{i_c} is ω-connected as well in colour i_c.

Proof. Let U be a non-principal ultrafilter on V with $\{N_G[F] : F \in [A]^{<\omega}\} \subset U$. For $i < r$ define $V_i = \{v \in V : N(v,i) \in U\}$ e.g. $d_c \restriction V_i \equiv i$, and let i_c be the unique element of $\{0, \dots, r-1\}$ with $V_{i_c} \in U$. It is not hard to check that this works. $\square$

The following was proved in [14]:

Lemma 1.3.6. *Suppose that $G = (V, E)$ is a countably infinite graph and c is an edge colouring. Suppose that $\{C_j : j < k\}$ is a finite family of subsets of V and that each C_j is ω-unseparable in some colour i_j. Moreover, for $j < k$ let $A_j \subseteq C_j$ be arbitrary subsets.*

Then we can find disjoint sets P_j so that

(a) *P_j is a path (either finite or one-way infinite) in colour i_j for all $j < k$,*

(b) *if A_j is infinite then so is $A_j \cap P_j$,*

(c) *$\bigcup\{P_j : j < k\} \supset \bigcup\{C_j : j < k\}$.*

Moreover, if a C_j is infinite then we can choose the first point of P_j freely from C_j.

Proof. Let $v_0, v_1, \dots$ be a (possibly finite) enumeration of $\bigcup\{C_j : j < k\}$.

For all the infinite C_j, fix distinct $x_j \in C_j$ as starting points for the P_js. We define disjoint finite paths $\{P_j^n : j < k\}$ by induction on $n \in \mathbb{N}$ so that

(i) P_j^n is a path of colour i_j with first point x_j,

(ii) P_j^{n+1} end extends P_j^n (as a path of colour i_j),

(iii) the last point of the path P_j^n is in C_j,

(iv) if A_j is infinite then the last point of P_j^{2n} is in A_j,

for all $j < k$, and

(v) if $v_n \notin \bigcup_{j<k} P_j^{2n}$ and $v_n \in C_j$ then v_n is the last point of P_j^{2n+1}.

It should be easy to carry out this induction applying that each A_j is infinitely linked in colour i_j. Finally, we let $P_j = \cup\{P_j^n : n \in \mathbb{N}\}$ for $j < k$ which finishes the proof. $\square$

The following, essentially proved in [78, Theorem 2], is now an immediate corollary:

Theorem 1.3.7. *Suppose that $G = (V, E)$ is a graph, $A \in [V]^\omega$ and $N_G[F]$ is infinite for all $F \in [A]^{<\omega}$. Then for any finite edge colouring of G we can cover A by finitely many disjoint monochromatic paths of different colours.*

Proof. Apply Lemma 1.3.5 to partition A into sets $\{A_i : i < r\}$ where A_i is ω-unseparable in colour i. Now Lemma 1.3.6 provides the cover by disjoint monochromatic paths of different colours. $\square$

This theorem provides an alternate proof to Rado's result which was the starting point of our investigations.

Theorem 1.3.8 (R. Rado, [78])**.** *Given a finite edge colouring of the graph $K_{\mathbb{N}}$, the vertices can be partition into disjoint paths of different colours.*

The following example shows that Observation 1.3.4 cannot be extended (word by word) to the uncountable case.

Example 1.3.9. *There is a graph G which contains no uncountable paths however $N_G[F]$ is uncountable for all finite $F \subseteq V(G)$.*

Proof. Take a partition of ω_1 into uncountable sets X_F with $F \in [\omega_1]^{<\omega}$. Let $G = (\omega_1, E)$ with

$$E = \{\{\alpha, \beta\} : \alpha \in F, \beta \in X_F \setminus (\max F + 1), F \in [\omega_1]^{<\omega}\}.$$

It is clear that $N_G[F]$ is uncountable for all finite $F \subseteq V(G)$ and $|N_G(\alpha) \cap \alpha| < \omega$ for all $\alpha < \omega_1$.

The following observation leads to a contradiction if G contains an uncountable path.

Observation 1.3.10. *If a graph $G = (\omega_1, E)$ contains a path of size ω_1 then there is a club $C \subset \omega_1$ so that for all $\alpha \in C$ there is $\beta \in C \setminus \alpha$ with*

$$\sup N_G(\beta) \cap \alpha = \alpha.$$

Indeed, take any countable elementary submodel M of $H(\omega_2)$ with $G, P \in M$ and let α denote the $<_P$-minimal element of $P \setminus M$. Note that $M \cap P$ is an initial segment of P and α must be a $<_P$-limit in P. Hence, the infinite set $N(\alpha) \cap \{\xi \in \omega_1 : \xi <_P \alpha\}$ is contained in $M \cap \omega_1 \subseteq \alpha$ which finishes the proof. □

However, every uncountable path contains large unseparable sets:

Observation 1.3.11. *If P is a path of order type ω_1 then $\{v \in P : |N_P(v)| = \omega_1\}$ is ω_1-unseparable in P.*

To abbreviate the formulation of certain result we introduce the following notation.

Definition 1.3.12. *Let G be a graph and $\mathfrak{F}$ be a class of graphs. We write*

$$G \sqsubset (\mathfrak{F})_{r,m}$$

if given any r-edge colouring $c : E(G) \to r$ the vertex set of G can be partitioned into m monochromatic elements of $\mathfrak{F}$.

We write

$$G \sqsubset (\mathfrak{F}, \mathfrak{F}, \ldots, \mathfrak{F})_r$$

if given any r-edge colouring $c : E(G) \to r$ the vertex set of G can be partitioned into r monochromatic elements of $\mathfrak{F}$ in distinct colours.

In particular, $G \sqsubset (\mathfrak{Path})_{r,m}$ holds if given any r-edge colouring c of G the vertex set of G can be partitioned into m monochromatic paths.

We write $\sqsubset^*$ instead of $\sqsubset$ if we can partition the vertex set apart from a finite set.

Using our new notation, Theorem 1.3.8 can be formulated as follows:

$$K_{\mathbb{N}} \sqsubset (\mathfrak{Path}, \ldots, \mathfrak{Path})_r.$$

1.4 The chromatic number and colouring number

We say that $f : V(G) \to \kappa$ is a good colouring of a graph G iff $f^{-1}(\alpha)$ is independent for all $\alpha < \kappa$.

Definition 1.4.1. *The* chromatic number *of a graph G, denoted by $Chr(G)$ is the smallest cardinal κ such that there is good colouring of G with κ colours.*

The following definition was introduced in [20]:

Definition 1.4.2. *The* colouring number *of a graph G is the least cardinal μ such that there is a well ordering $\prec$ of $V(G)$ such that $\{v \in N_G(w) : v \prec w\}$ has size $< \mu$ for all $w \in V(G)$.*

An easy argument shows

Fact 1.4.3. *If $Col(G) = \mu$ then $Chr(G) \leq \mu$.*

However, $Col(K_{\mu,\mu}) = \mu$ while $Chr(K_{\mu,\mu}) = 2$ for each infinite μ. Also, let us mention that one can always find a well ordering of order type $|V(G)|$ witnessing that G has colouring number $Col(G)$ [20].

In Chapter 4, we look into the question if having large chromatic or colouring number implies that the graph has certain obligatory subgraphs. While we will not deal with subtle differences between the chromatic and colouring number, it is worth mentioning that these two graph parameters are surprisingly different. An excellent example of this phenomena is a consequence of Shelah's singular compactness theorem:

Theorem 1.4.4 ([87, Conclusion 2.3.])**.** *Suppose that $|V(G)| = \lambda > cf(\lambda)$ and $Col(G) > \omega$. Then there is a subgraph G' of G of size $< \lambda$ such that $Col(G') > \omega$.*

This theorem fails if one replaces colouring number by chromatic number (as shown by P. Komjáth [57] and S. Shelah [90]).

1.5 Trees

A set theoretic *tree* $(T, \leq)$ is a partially ordered set such that

$$t^{\downarrow} = \{s \in T : s < t\}$$

is well ordered for all $t \in T$. Note that this notion of a tree has little to do with *graph theoretic trees* i.e. connected graphs without circles; in this paper, by a tree we will always mean a set theoretic tree. Every tree admits a *height* function: $ht(t)$ denotes the order type of $t^{\downarrow}$ for $t \in T$. The *height of the tree* T is $\sup\{ht(t) : t \in T\}$.

Definition 1.5.1. *Let $G(T)$ denote the comparability graph of a tree T i.e. the set of vertices of $G(T)$ is T and $\{x, y\} \in [T]^2$ is an edge iff $x \leq y$ or $y \leq x$.*

Note that $A \subset T$ is independent in $G(T)$ iff A is an antichain in T thus $Chr(G(T)) \leq \omega$ iff T is a special tree. We will be interested in constructing subgraphs of a graph $G(T)$ in order to solve problems concerning connectivity and chromatic number.

Our tools in Chapter 6 are based on certain non special trees T (so $Chr(G(T)) > \omega$) which contain *no uncountable chains*. An example of such a tree is any non-special Aronszajn tree however there are examples purely in ZFC as well.

Let $\sigma\mathbb{Q} = \{s \subseteq \mathbb{Q} : s \text{ is bounded and well ordered in } \mathbb{Q}\}$ with $s \leq t$ iff s is an initial segment of t. $\sigma\mathbb{Q}$ was first studied by D. Kurepa in connection with Souslin's problem [68]. It is clear that $\sigma\mathbb{Q}$ contains no chains of size ω_1 and has size 2^ω and height ω_1. Moreover

Theorem 1.5.2 ([68])**.** *$\sigma\mathbb{Q}$ is non special.*

Hence $Chr(G(\sigma\mathbb{Q})) = \omega_1$.

Another classical example is defined as follows:

$$T(S) = \{t \subset S : t \text{ is closed}\}$$

with $s \leq t$ iff s is an initial segment of t where $S \subseteq \omega_1$. Note that $T(S)$ has height $\leq \omega_1$, size $\leq 2^\omega$ and contains no uncountable chains if S does not contain a club, i.e. S is costationary.

The importance of trees of the form $T(S)$ was realized by S. Todorcevic [98]. Todorcevic used the trees $T(S)$ to define a family of continuums $C(S)$ with fascinating properties. The tree $T(S)$ as a forcing notion was first defined by Jensen (see the end of Section 9 in [99]) and also appears in [7, 11].

Trees of the form $T(S)$ have the following nice property: $T(S)$ has *no branching at limit levels* i.e. $t^\downarrow = s^\downarrow$ implies $t = s$ for all limit elements $t, s \in T(S)$. This is proved using that every element of $T(S)$ is a *closed* subset of ω_1.

Let us cite some facts from [99]. Let $T \otimes T'$ denote $\{(t,t') \in T \times T : ht(t^\downarrow) = ht(t'^\downarrow)\}$ with the pointwise ordering.

Theorem 1.5.3 ([99, Theorem 3.4])**.** *(i) $T(S)$ is special iff S is nonstationary in ω_1;*

(ii) $T(S) \otimes T(S')$ is special iff $S \cap S'$ is nonstationary in ω_1.

Hence, if S is stationary, costationary then $T(S)$ has chromatic number ω_1. A tree T is called *Baire* iff the intersection of countably many dense final parts is still dense in T.

Theorem 1.5.4 ([99, Theorem 3.7])**.** *(i) $T(S)$ is Baire iff S is stationary in ω_1;*

(ii) $T(S) \otimes T(S')$ is Baire iff $S \cap S'$ is stationary in ω_1.

Finally, different $S \subseteq \omega_1$ define rather different trees $T(S)$ and hence different comparability graphs. The following was essentially proved in [99, Theorem 5.1]:

Theorem 1.5.5. *Suppose that $S_0, S_1 \subseteq \omega_1$ and $G_i = G(T(S_i))$ for $i < 2$. If there is a 1-1 homomorphism $f : G_0 \to G_1$ then $S_0 \setminus S_1$ is nonstationary.*

Proof. Suppose that $S_0 \setminus S_1$ is stationary and take a countable elementary submodel $M \prec H(\omega_2)$ so that $S_0, S_1, f \in M$ and $\delta = M \cap \omega_1 \in S_0 \setminus S_1$.

Claim 1.5.6. *For every $s \in T(S_0)$ and $\varepsilon \in \omega_1$ there is $s' \geq s$ in $T(S_0)$ such that* $\max f(s'), \max(s') \geq \varepsilon$.

Indeed, note that $G_1 \restriction \{t \in T(S_1) : \max(t) \leq \varepsilon\}$ has countable chromatic number and hence so does $G_0 \restriction f^{-1}\{t \in T(S_1) : \max(t) \leq \varepsilon\}$. Hence

$$\{s' \in T(S_0) : s' \geq s, \max(s') \geq \varepsilon\} \setminus f^{-1}\{t \in T(S_1) : \max(t) \leq \varepsilon\} \neq \emptyset$$

as $G_0 \restriction \{s' \in T(S_0) : s' \geq s, \max(s') \geq \varepsilon\}$ is not countably chromatic (as S_0 is stationary).

Now, apply the above claim inductively in M to find an increasing sequence $\{s_n : n \in \omega\} \subseteq T(S_0) \cap M$ such that $\{f(s_n) : n \in \omega\} \subseteq T(S_1) \cap M$ is increasing as well and

$$\sup\{\max(s_n) : n \in \omega\} = \sup\{\max f(s_n) : n \in \omega\} = \delta.$$

Let $s = \bigcup\{s_n : n \in \omega\} \cup \{\delta\}$ and note that $s \in T(S_0)$. Let $t = f(s) \in T(S_1)$ and note that $t \notin M$ as $s \notin M$ and $f^{-1}\{s\} = \{t\}$. Hence we have $f(s_n) \subset t$ for all n (as f preserves the edge relation) and hence $\delta \in t$ as t is closed in ω_1. However $\delta \notin S_1$ which contradicts $t \in T(S_1)$.

$\square$

We will use trees of the form $T(S)$ throughout Chapter 6.

We mention that the class of trees with no uncountable branches were studied in [13] more recently.

1.6 Elementary submodels

The first application of elementary submodels in combinatorial setting was most likely due to S. G. Simpson [92] from 1970 who presented a new proof to the classical Erdős-Rado partition relation. Simpson already mentioned that *"one can give similar proofs for several other known theorems of combinatorial set theory ..."*

The literature contains several well written introductions to (chains of) elementary submodels and their applications in topology and combinatorics; see the papers [12, 30, 94], the book [50] or the presentation [95]. Nowadays every other proof in set theory and general topology uses elementary submodels and we will hence assume basic familiarity with this tool. Nonetheless, we include a very short, and over-simplified, introduction: we will work with elementary submodels M of $H(\Theta)$ (sets of hereditary cardinality $< \Theta$ for some large enough cardinal Θ). $H(\Theta)$ captures a large fragment of the set theoretic universe (i.e. almost all of ZFC is satisfied) and M being an elementary submodel means that a formula ϕ with parameters from M is true in M iff it is true in $H(\Theta)$.

How are elementary submodels useful? If a structure $\mathcal{X}$ of arbitrary size is intersected with say a countable elementary submodel M so that $\mathcal{X} \in M$ then the structure $\mathcal{X} \cap M$ will be very similar to $\mathcal{X}$ but has countable size; we say that properties of $\mathcal{X}$ *reflect* to $M \cap \mathcal{X}$. It is easy to imagine that such a construction is useful in many situations.

Why are there always elementary submodels which have all the parameters we need for a certain proof? The downward Löwenheim-Skolem theorem says that whenever $\mathcal{A} \subset H(\Theta)$ is countable then we can find a countable elementary submodel M of $H(\Theta)$ so that $\mathcal{A} \subset M$. i.e. M contains everything relevant to our particular situation. We regularly use the following

Fact 1.6.1. *Suppose that M is an elementary submodel of $H(\Theta)$ and $X \in M$. If X is countable then $X \subseteq M$ or equivalently, if $X \setminus M$ is non-empty then X is uncountable.*

In Chapter 3, we make use of the following

Definition 1.6.2. *A* nice κ-chain of elementary submodels *is an increasing sequence $(M_\alpha)_{\alpha<cf(\kappa)}$ of elementary submodels of $H(\Theta)$ (for some large enough cardinal Θ) with $|M_\alpha| = \kappa_\alpha$ so that $M_0 = \emptyset$ and*

1. *$\kappa_{\alpha+1}$ is a subset and element of $M_{\alpha+1}$,*
2. *M_α is a subset and element of M_β if $\alpha < \beta < cf(\kappa)$,*

3. $M_\beta = \bigcup\{M_\alpha : \alpha < \beta\}$ *for any limit* $\beta < cf(\kappa)$,

4. *if* κ *is a limit cardinal then* $(\kappa_\alpha)_{\alpha<cf(\kappa)}$ *strictly increases to* κ *and* $cf(\kappa_{\alpha+1}) = \kappa_{\alpha+1}$ *for every* $\alpha < cf(\kappa)$,

5. *if* κ *is regular then* $M_\alpha \cap \kappa \in \kappa$, *if* $\kappa = \lambda^+$ *then* $\kappa_\alpha = \lambda$ *for all* $\alpha < \kappa$.

Observation 1.6.3. *Let* $G = (V, E)$ *be a graph and* $A \subseteq V$ κ*-unseparable. Suppose that* $(M_\alpha)_{\alpha<cf(\kappa)}$ *is a nice* κ*-chain of elementary submodels covering* A *so that* $A, G \in M_1$. *Then* $A \cap (M_{\alpha+1} \setminus M_\alpha)$ *is* $|M_{\alpha+1}|$*-unseparable in* $V \cap (M_{\alpha+1} \setminus M_\alpha)$ *for all* $\alpha < cf(\kappa)$.

Proof. Fix $\alpha < cf(\kappa)$ and two vertices $u, v \in A \cap (M_{\alpha+1} \setminus M_\alpha)$. As $M_{\alpha+1} \models A$ is $\kappa^+_{\alpha+1}$-unseparable, we can find a $\kappa^+_{\alpha+1}$-sequence $\mathcal{P} \in M_{\alpha+1}$ of disjoint finite path from u to v. As $M_\alpha \in M_{\alpha+1}$ and $|M_\alpha| < \kappa^+_{\alpha+1}$, we can suppose that each path in $\mathcal{P}$ is disjoint from M_α. Now, using $\kappa_{\alpha+1} \subseteq M_{\alpha+1}$, we have that

$$\mathrm{ran}(\mathcal{P} \upharpoonright \kappa_{\alpha+1}) \subseteq M_{\alpha+1}.$$

As each path in $\mathcal{P}$ is finite, we actually have $Q \subseteq V \cap (M_{\alpha+1} \setminus M_\alpha)$ for $Q \in \mathcal{P} \upharpoonright \kappa_{\alpha+1}$ which finishes the proof. □

In Chapter 5, we use special sequences of countable elementary submodels called Davies-trees or ω_1-approximation sequences; we postpone the definition to Section 5.2.

Chapter 2

Path decompositions of countable graphs and hypergraphs

2.1 A short history of path decomposition problems

P. Erdős proved the following result in the 70's [1]:

Theorem 2.1.1 (Erdős [78, Theorem 1]). *Every 2-edge coloured copy of the complete graph on $\mathbb{N}$ can be partitioned into two monochromatic paths of different colours.*

In 1978, R. Rado generalized Erdős's argument in the following way:

Theorem 2.1.2 ([78, Theorem 2]). *Suppose that $G = (V, E)$ is an infinite directed graph, $A \in [V]^{\leq\omega}$ and $|\{y \in V : (x, y) \notin E\}| < |V|$ for every $x \in A$. Then given any edge colouring c of G there is $J \subseteq \operatorname{ran}(c)$ and disjoint 1-1 sequences of points $\{x_j(\nu) : \nu < m_j\}$ for $j \in J$ such that*

1. *m_j is either finite and odd or $m_j = \omega$,*
2. *A is covered by $\bigcup_{j\in J}\{x_j(\nu) : \nu < m_j\}$, and*
3. *if $j \in J$ and $\nu < m_j$ is odd then*

$$(x_j(\nu - 1), x_j(\nu)), (x_j(\nu + 1), x_j(\nu)) \in c^{-1}(j).$$

The following theorem is an easy corollary now:

Theorem 2.1.3 ([78]). *If the edges of the complete graph on $\mathbb{N}$ are coloured with finitely many colours then the vertices can be partitioned into disjoint monochromatic paths of different colours.*

This result was the starting point of several papers in the past which dealt with the same problem either on finite or countably infinite graphs. It is an easy exercise to prove Theorem 2.1.1 for finite complete graphs. Also, there are examples due to K. Heinrich[2] of r-edge coloured finite complete graphs

[1] Appeared in [78] through private communication with R. Rado.
[2] Private communication with A. Gyárfás [36]

which cannot be partitioned into r monochromatic paths of *different colours*. Hence Theorem 2.1.3 does not extend to the finite case word by word.

In [34], Gyárfas looked at the path decomposition question from an algorithmic viewpoint. Given a 2-edge coloured copy of K_n, one can find (1) a Hamiltonian cycle which is the union of a red and blue path, or (2) a cover of K_n by a red cycle and a blue cycle with at most one common vertex, both in $O(n)$ steps.

The question if Theorem 2.1.3 extends to finite complete graphs was first asked by Gyárfas [35] in 1989; in the same paper, he presents the bound

$$f(r) = 2r^2(\binom{r+1}{2} + 1) + 1$$

for the number of monochromatic paths needed to cover an r-edge coloured finite complete graph.

In the meanwhile, there was significant work done in order to prove a closely related conjecture due to Lehel (first reference in [3]): every two edge coloured finite complete graph can be partitioned into two disjoint cycles of different colours. Lehel's conjecture was justified for graphs of large order by T. Luczak et al. [69] in 1998 and a complete solution was provided by S. Bessy and S. Thomassé [8] in 2010.

The monochromatic cycle partition problem on finite complete graphs was studied for more colours as well in several papers [19, 46, 38]. In [19], P. Erdős, A. Gyárfás and L. Pyber made the following conjecture: every r-edge coloured finite complete graph can be partitioned into r disjoint monochromatic cycles. The currently known strongest positive result is the following

Theorem 2.1.4 ([38]). *For every integer $r \geq 2$ there is $n_0(r) \in \mathbb{N}$ such that if $n \geq n_0(r)$ then every r-edge coloured copy of K_n can be partitioned into at most $100r\log(r)$ monochromatic cycles.*

However, recently A. Pokrovskiy [77] proved

Theorem 2.1.5 ([77, Theorem 1.4]). *Suppose that $r \geq 3$. There are infinitely many r-edge coloured finite complete graphs which cannot be partitioned into r disjoint monochromatic cycles.*

However, an infinite version of the Erdős-Gyárfás-Pyber conjecture is true (see Theorem 2.2.2(2)). Also, the following version of the Erdős-Gyárfás-Pyber conjecture could still be true:

Conjecture 2.1.6. *Every r-edge coloured finite complete graph is covered by r monochromatic cycles.*

Let us return to the question of finding path decompositions of finite complete graphs. In general, Theorem 2.1.4 gives the best known bound for the number of paths needed in a monochromatic path partition. For a small number of colours r other than 2, the only result we have is the recent

Theorem 2.1.7 ([77, Theorem 1.5]). *Every 3-edge coloured finite complete graph can be partitioned into 3 disjoint monochromatic paths.*

In particular, the following question is still open:

Problem 2.1.8. *Does every 4-edge coloured finite complete graph admit a partition into 4 monochromatic paths?*

It is quite natural at this point to look at path/cycle decompositions of edge coloured non-complete graphs and edge coloured complete hypergraphs as well. Monochromatic cycle partitions of complete bipartite graphs are investigated in [46]. In [77], Pokrovskiy shows that every two edge coloured $K_{n,n}$, can be partitioned into 3 monochromatic paths and the following conjecture is stated:

Conjecture 2.1.9 ([77, Conjecture 4.5])**.** *Suppose that the edges of $K_{n,n}$, are coloured with r colours. Then there is a partition of $K_{n,n}$, into $2r-1$ disjoint monochromatic paths.*

It will be rather easy to show that this conjecture holds for the graph $K_{\omega,\omega}$ (see Theorem 2.4.1).

In [37], the authors look at path decompositions of a 2-edge coloured graph G on n vertices with $n/2$ edges missing. In [83], G. Sárközy looks at cycle partitions of edge coloured graphs G with given independence number $\alpha(G)$ (recall that $\alpha(G)$ is the size of the largest independent set in G). This line of research was carried on in [4]: it is proved that every 2-edge coloured finite graph G can be partitioned into $2\alpha(G)$ monochromatic cycles. In the same paper, further cycle decomposition results are proved for graphs with given minimal degree.

Path and cycle partition problems on hypergraphs were investigated by Gyárfás and Sárközy [40, 39]. Let us mention two results here:

Theorem 2.1.10 ([39, Theorem 1.])**.** *For all integers $r > 1, k > 2, \alpha > k-1$ there exists a constant $c = c(r, k, \alpha)$ such that for every r-edge colouring of a k-uniform hypergraph H with independence number $\alpha(H) = \alpha$ there is a partition of the vertices into at most $c(r, k, \alpha)$ disjoint monochromatic loose cycles.*

Theorem 2.1.11 ([40, Theorem 3.])**.** *Suppose that the edges of a countably infinite complete k-uniform hypergraph are coloured with r colours. Then the vertex set can be partitioned into monochromatic finite or one-way infinite loose paths of distinct colours.*

Our goal in Section 2.2 is to prove a stronger version of the latter theorem.

The interested reader may look into the survey [51] by M. Kano and X. Li on problems and results on finding monochromatic and heterochromatic subgraphs of edge-coloured graphs.

2.2 Partitions of hypergraphs

In this section, we aim to improve Theorem 2.1.11. Let $k \in \mathbb{N} \setminus \{0\}$.

Definition 2.2.1. *A* loose path *in a k-uniform hypergraph is a finite or one-way infinite sequence of edges, $e_1, e_2, \dots$ such that $|e_i \cap e_{i+1}| = 1$ for all i, and $e_i \cap e_j = \emptyset$ for all i, j with $i + 1 < j$.*

A tight path *in a k-uniform hypergraph is a finite or one-way infinite sequence of distinct vertices such that every set of k consecutive vertices forms an edge.*

Remark. Occasionally, we will refer to loose and tight *cycles* and *two-way infinite paths* as well, with the obvious analogous definitions.

In the introduction of [40], the authors asked if in Theorem 2.1.11 one can find a partition into tight paths instead of loose ones. We prove the following:

Theorem 2.2.2. *Suppose that the edges of a countably infinite complete k-uniform hypergraph are coloured with r colours. Then*

(1) the vertex set can be partitioned into monochromatic finite or one-way infinite tight paths of distinct colours,

(2) the vertex set can be partitioned into monochromatic tight cycles and two-way infinite tight paths of distinct colours.

Proof. (1) Note that the case of $k = 2$ is Rado's Theorem 1.3.8 above; we will imitate his original proof here.

Let $c : [\mathbb{N}]^k \to \{0, \dots, r-1\}$. A set $T \subset \{0, \dots, r-1\}$ of colours is called *perfect* iff there are disjoint finite subsets $\{P_t : t \in T\}$ of $\mathbb{N}$ and an infinite set $A \subset \mathbb{N} \setminus \bigcup_{t\in T} P_t$ such that for all $t \in T$

(a) P_t is a tight path in colour t,

(b) if $1 \le i < k$ and x is the set of the last i vertices from the tight path P_t and $y \in [A]^{k-i}$, then $c(x \cup y) = t$.

Since $\emptyset$ is perfect, we can consider a perfect set T of colours with maximal number of elements.

Claim 2.2.3. *If the vertex disjoint finite tight paths $\{P_t : t \in T\}$ and the infinite set A satisfy (a) and (b) then for all $v \in \mathbb{N} \setminus \bigcup_{t\in T} P_t$ there is a colour $t' \in T$, a finite sequence $v_1, v_2, \dots, v_{k-1}$ from A, and an infinite set $A' \subset A$ such that the tight paths*

$$\{P_t : t \in T \setminus \{t'\}\} \cup \{P_{t'}{}^\frown(v_1, v_2, \dots, v_{k-1}, v)\}$$

and A' satisfy (a) and (b) as well.

Proof of the Claim. Define a new colouring $d : [A]^{k-1} \to \{0, \dots, r-1\}$ by the formula $d(x) = c(x \cup \{v\})$. By Ramsey's Theorem, there is an infinite d-homogeneous set $B \subset A$ in some colour t'. Then $t' \in T$, since otherwise $T \cup \{t'\}$ would be a bigger perfect set witnessed by $P_{t'} = \{v\}$,$\{P_t : t \in T\}$ and B.

Now pick distinct $v_1, v_2, \dots, v_{k-1}$ from B and let $A' = B \setminus \{v_1, \dots, v_{k-1}, v\}$. $\square$

Finally, by applying the claim repeatedly, we can cover the vertices with $|T|$ tight paths of distinct colours.

(2) Let $c : [\mathbb{N}]^k \to \{0, \dots, r-1\}$. Write $V_{-1} = \mathbb{N}$. Using Ramsey's Theorem, by induction on $n \in \mathbb{N}$ choose $d(n) < r$ and $V_n \in [V_{n-1}]^{\mathbb{N}}$ such that

$$c(\{n\} \cup O) = d(n) \text{ for all } O \in [V_n]^{k-1}. \tag{2.1}$$

For $i < r$ let

$$A_i = \{n \in \mathbb{N} : d(n) = i\}. \tag{2.2}$$

Let $K = \{i < r : A_i \text{ is finite}\}$. By induction on $i \in K$ we will define tight cycles $\{P_i : i \in K\}$ such that

$$\bigcup_{i'<i, i'\in K} A_{i'} \subseteq \bigcup_{i'<i, i'\in K} P_{i'}$$

while some of the P_i's might be empty.

Assume that $\{P_{i'} : i' < i, i' \in K\}$ is defined and suppose $i \in K$. Enumerate $A_i \setminus \bigcup_{i'<i,i'\in K} P_{i'}$ as $\{x_i^j : j < t\}$.

Choose disjoint $k-1$ element sets

$$Y_i^j \subseteq \bigcap_{j<t} V_{x_i^j} \setminus \bigcup_{i'<i, i'\in K} P_{i'} \text{ for } j < t. \tag{2.3}$$

Consider an ordering $\prec_i$ on $P_i = \{x_i^j : j < t\} \cup \bigcup_{j<t} Y_i^j$ such that

$$x_i^0 \prec_i Y_i^0 \prec_i x_i^1 \prec_i Y_i^1 \prec_i \cdots \prec_i x_i^{t-1} \prec_i Y_i^{t-1}.$$

Then $\prec_i$ witnesses that P_i is a tight cycle in colour i.

Now, let

$$P = \bigcup_{i \in K} P_i$$

and for each $i \in \{0, \ldots, r-1\} \setminus K$ we define a 2-way infinite tight path P_i as follows.

By induction, for every integer $z \in \mathbb{Z}$ and $i \in \{0, \ldots, r-1\} \setminus K$ choose disjoint sets $\{x_i^z\} \in [A_i \setminus P]^1$ and $Y_i^z \in [\mathbb{N} \setminus P]^{k-1}$ such that

$$Y_i^z \subset V_{x_i^z} \cap V_{x_i^{z+1}}$$

and

$$\bigcup_{i \in \{0,\ldots,r-1\} \setminus K} A_i \subset P \cup \bigcup \{\{x_i^z\}, Y_i^z : i \in \{0, \ldots, r-1\} \setminus K, z \in \mathbb{Z}\}.$$

Consider an ordering $\prec_i$ on $P_i = \{x_i^z : z \in \mathbb{Z}\} \cup \bigcup_{z \in \mathbb{Z}} Y_i^z$ such that

$$\ldots \prec_i Y_i^{-2} \prec_i x_i^{-1} \prec_i Y_i^{-1} \prec_i x_i^0 \prec_i Y_i^0 \prec_i x_i^1 \prec_i Y_i^1 \prec_i \ldots$$

Then $\prec_i$ witnesses that P_i is a 2-way infinite tight path in colour i. $\square$

2.3 Covers by k^{th} powers of paths

We will be interested in partitioning an edge coloured copy of $K_{\mathbb{N}}$ into finitely many monochromatic k^{th} *powers of paths*:

Definition 2.3.1. *Suppose that $G = (V, E)$ is a graph and $k \in \mathbb{N} \setminus \{0\}$. The k^{th}* power of G *is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a finite path of length $\leq k$ from v to w.*

Figure 2.1 below shows the first three powers of a path.

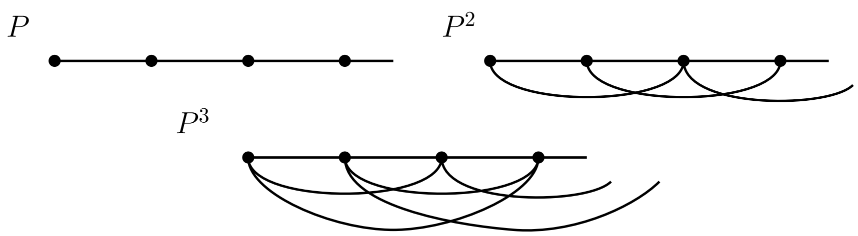


Figure 2.1: Powers of paths.

We will investigate this new decomposition problem by introducing the following game.

Definition 2.3.2. *Assume that H is a graph, $W \subset V(H)$ and $k \in \mathbb{N}$. The game $\mathfrak{G}_k(H, W)$ is played by two players, Adam and Bob, as follows. The players choose disjoint finite subsets of $V(H)$ alternately:*

$$A_0, B_0, A_1, B_1, \ldots$$

Bob wins the game $\mathfrak{G}_k(H, W)$ *iff*

(A) $W \subset \bigcup_{i\in\mathbb{N}} A_i \cup B_i$*, and*

(B) $H[\bigcup_{i\in\mathbb{N}} B_i]$ *contains the* k^{th} *power of a (finite or one way infinite) Hamiltonian path (that is, a path covering all the vertices).*

For $k = 1$, we have the following

Observation 2.3.3. *If* $H = (V, E)$ *is a countable graph and* $W \subset V$ *then the following are equivalent:*

1. W *is* ω*-unseparable,*
2. *Bob wins* $\mathfrak{G}_1(H, W)$*.*

Proof. (1) $\Rightarrow$ (2): By our assumption, Bob can always connect an uncovered point of W to the end-point of the previously constructed path while avoiding vertices played so far. This shows the existence of a winning strategy for Bob.

(2) $\Rightarrow$ (1): Fix any two distinct points $v, w \in W$ and a finite set $F \subset V \setminus \{v, w\}$. Let Adam start with $A_0 = F$ and continue with $A_i = \emptyset$; the Hamiltonian path P constructed by Bob's strategy will go through a and b while $P \cap F = \emptyset$. □

Now, we show how to produce a partition of the vertices into k^{th} powers of paths using winning strategies of Bob:

Lemma 2.3.4. *Suppose that* $H = (V, E)$*,* $V = \bigcup\{W_i : i < M\}$ *with* $M \in \mathbb{N}$ *and let* $H_i = (V, E_i)$ *for some* $E_i \subset E$*. If Bob wins* $\mathfrak{G}_k(H_i, W_i)$ *for all* $i < M$ *then* V *can be partitioned into* $\{P_i : i < M\}$ *so that* P_i *is a* k^{th} *power of a path in* H_i*.*

Proof. We will conduct M games simultaneously as follows: the plays of Adam and Bob in the i^{th} game will be denoted by $A^i_0, B^i_0, A^i_1, B^i_1, \dots$ for $i < M$. Let σ^i denote the winning strategy for Bob in $\mathfrak{G}_k(H_i, W_i)$, that is, if we set $B^i_n = \sigma^i(A^i_0, B^i_0, \dots, A^i_n)$ then Bob wins the game.

Now, we define A^i_n, B^i_n by induction using the lexicographical ordering $<_{lex}$ on $\{(n, i) : n \in \mathbb{N}, i < M\}$. First, let $A^0_0 = \emptyset$ and $B^0_0 = \sigma^0(A^0_0)$. In general, assume that A^j_m and B^j_m are defined for $(m, j) <_{lex} (n, i)$, and we let

$$A^i_n = \bigcup\{B^j_m : (m, j) <_{lex} (n, i)\} \setminus \Big(\bigcup\{A^i_m, B^i_m : m < n\}\Big) \tag{2.4}$$

and

$$B^i_n = \sigma^i(A^i_0, B^i_0, \dots, A^i_n).$$

One easily checks that the above defined plays are valid; indeed, for a fix $i < M$ the finite sets $\{A^i_n, B^i_n : n \in \mathbb{N}\}$ defined above are disjoint.

Next, let $P_i = \bigcup\{B^i_n : n \in \mathbb{N}\}$ for $i < M$. As Bob wins the i^{th} game we have that P_i is a k^{th} power of path in H_i. Note that $P_i \cap P_j = \emptyset$ if $i \neq j < M$. Indeed, if $(m, j) <_{lex} (n, i)$, then

$$B^i_n \cap B^j_m \subset B^i_n \cap (A^i_n \cup \Big(\bigcup\{A^i_m, B^i_m : m < n\}\Big) = \emptyset$$

by (2.4).

To finish the proof, we prove

$$V = \{P_i : i < M\}. \tag{2.5}$$

Indeed, first note that $W_i \subset \bigcup_{n\in\mathbb{N}} A^i_n \cup B^i_n$ as Bob wins the i^{th} game and hence

$$V = \bigcup_{n\in\mathbb{N}, i<M} A^i_n \cup B^i_n.$$

Second, by (2.4), we have

$$A^i_n \subset \bigcup\{B^j_m : (m,j) <_{lex} (n,i)\}$$

and so

$$\bigcup_{n\in\mathbb{N}, i<M} A^i_n \subset \bigcup_{n\in\mathbb{N}, i<M} B^i_n$$

and hence $V = \{P_i : i < M\}$. □

The next theorem provides conditions under which Bob has a winning strategy:

Theorem 2.3.5. *Assume that H is a countably infinite graph, $W \subset V(H)$ is non-empty and $k \in \mathbb{N}$. If there are subsets $W_0, \ldots, W_k$ of $V(H)$ such that $W_0 = W$ and*

$$W_{j+1} \cap N_H[F] \text{ is infinite for each } j < k \text{ and finite } F \subset \bigcup\nolimits_{i\le j} W_i \tag{2.6}$$

then Bob wins $\mathfrak{G}_k(H, W)$.

Proof. We can assume that $\mathrm{V}(H) = \mathbb{N}$.

Consider first the easy case when W_0 is finite. Adam plays a finite set A_0 in the first round. Write $N = |W_0 \setminus A_0|$. Let Bob play $B_0 = W_0 \setminus A_0 = \{b_{n,0} : n < N\}$. In the j^{th} round for $1 \le j \le k$, let Bob play an N-element set

$$B_j = \{b_{n,j} : n < N\} \subset W_j \cap N_H\big[\bigcup\nolimits_{i<j} B_i\big] \tag{2.7}$$

which avoids all previous choices, i.e. $B_j \cap \bigcup\{A_{i'}, B_i : i' \le j, i < j\} = \emptyset$. For $j > k$ let Bob play $B_j = \emptyset$.

We claim that

(A) $W_0 \subseteq \bigcup\{A_n, B_n : n \in \mathbb{N}\}$, and

(B) $P = \{b_{n,j} : n < N, j \le k\}$ is the k^{th}-power of a path.

(A) is clear because $W_0 \subseteq A_0 \cup B_0$.

To check (B) consider the lexicographical order of the indexes. Let $(m,i) \ne (n,j) \in \{0,\ldots,N-1\} \times \{0,\ldots,k\}$. Then $b_{m,i}$ and $b_{n,j}$ are the $((k+1)m+i)^{th}$ and $((k+1)n+j)^{th}$ elements, respectively, in the lexicographical order.

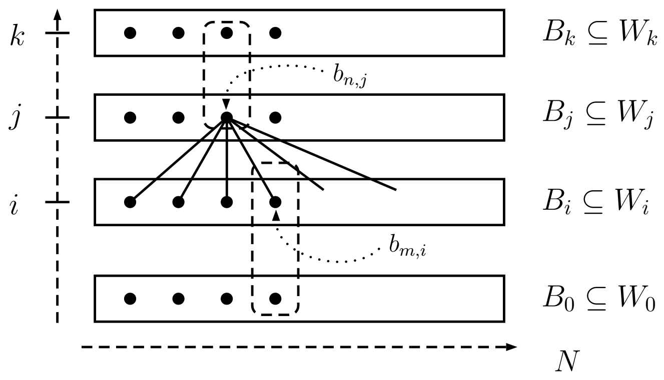


Figure 2.2: $b_{n,j}$ and its k successors.

Assume that $|((k+1)m+i)-((k+1)n+j)| \leq k$; then $i \neq j$ and, without loss of generality, we can suppose that $i < j$. Then we have $b_{m,i} \in \bigcup_{i'<j} B_{i'}$, so $b_{n,j} \in N_H(b_{m,i})$ by (2.7). In other words, $\{b_{m,i}, b_{n,j}\}$ is an edge in H which yields (B).

Consider next the case when W_0 is infinite; let us outline the idea first in the case when $k=2$. Bob will play one element sets at each step and aims to build a one-way infinite square of a path following the lexicographical ordering on $\mathbb{N} \times \{0,1,2\}$. However, he picks the vertices in a different order, denoted by $\trianglelefteq$ later, which is demonstrated in Figure 2.3.

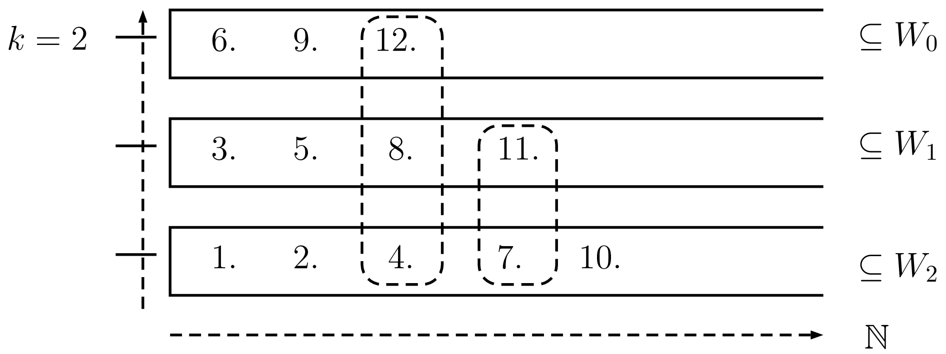


Figure 2.3: The two orderings.

This way Bob makes sure that when he chooses the 12^{th} element he already picked its two successors (in the 7^{th} and 11^{th} plays) and two predecessors (in the 8^{th} and 4^{th} plays) in the lexicographical ordering, hence we can ensure the edge relations here.

Now, we define the strategy more precisely. In each round Bob will pick a single element $b_{n,j}$ for some $(n,j) \in \mathbb{N} \times \{0,1,\ldots,k\}$ such that $\{b_{n,j} : (n,j) \in \mathbb{N} \times \{0,1,\ldots,k\}\}$ will be the k^{th} power of a path in the lexicographical order of $\mathbb{N} \times \{0,1,\ldots,k\}$.

As we said earlier, Bob will not choose the points $b_{n,j}$ in the lexicographical order of $\mathbb{N} \times \{0,1,\ldots,k\}$, i.e. typically the $((k+1)n+j)^{th}$ move of Bob, denoted by $B_{(k+1)n+j}$, is *not* $\{b_{n,j}\}$.

To describe Bob's strategy we should define another order on $\mathbb{N} \times \{0, 1, \ldots, k\}$ as follows:

$$(m, i) \trianglelefteq (n, j) \text{ iff } (m + i < n + j) \text{ or } (m + i = n + j \text{ and } i \leq j).$$

Write $(m, i) \triangleleft (n, j)$ iff $(m, i) \trianglelefteq (n, j)$ and $(m, i) \neq (n, j)$. Clearly every (n, j) has just finitely many $\triangleleft$-predecessors. Let $f(\ell)$ denote the ℓ^{th} element of $\mathbb{N} \times \{0, 1, \ldots, k\}$ in the order $\triangleleft$.

Bob will choose $B_\ell = \{b_{f(\ell)}\}$ in the ℓ^{th} round as follows: if $f(\ell) = (n, j)$, then

(a) if $j = 0$ then

$$b_{n,j} = \min\Big(W_0 \setminus (\bigcup_{s \leq \ell} A_s \cup \bigcup_{t < \ell} B_t)\Big); \tag{2.8}$$

(b) if $j > 0$ then

$$b_{n,j} \in W_j \cap N_H\big[\{b_{m,i} : (m, i) \triangleleft (n, j), i < j\}\big]. \tag{2.9}$$

Bob can choose a suitable $b_{n,j}$ by (2.6) as $\{b_{m,i} : (m, i) \triangleleft (n, j), i < j\}$ is a finite subset of $\bigcup_{i<j} W_i$.

We claim that

(A) $W_0 \subseteq \bigcup\{A_n, B_n : n \in \mathbb{N}\}$, and

(B) $P = \{b_{n,j} : n \in \mathbb{N}, j \leq k\}$ is the k^{th}-power of a path.

(A) is clear because in (2.8) we chose the minimal possible element.

Let $(m, i) \neq (n, j) \in \mathbb{N} \times \{0, \ldots, k\}$. Then $b_{m,i}$ and $b_{n,j}$ are the $((k+1)m+i)^{th}$ and $((k+1)n+j)^{th}$ elements, respectively, in the lexicographical order. Assume that $|((k+1)m+i) - ((k+1)n+j)| \leq k$. Then $i \neq j$ and $|m - n| \leq 1$.

Without loss of generality, we can assume that $i < j$. Then $|m - n| \leq 1$ implies $m + i \leq n + j$ and hence $(m, i) \triangleleft (n, j)$. Since $i < j$ as well, $b_{n,j} \in N_H(b_{m,i})$ must hold by (2.9). In other words, $\{b_{m,i}, b_{n,j}\}$ is an edge in H which yields (B). $\square$

We arrive at one of our main results:

Theorem 2.3.6. *For all positive natural numbers k, r and an r-edge colouring of $K_\mathbb{N}$ the vertices can be covered by $\leq r^{(k-1)r+1}$ one-way infinite disjoint monochromatic k^{th} powers of paths and a finite set.*

Proof. The set of sequences of length m (at most m, respectively) whose members are from a set X is denoted by X^m ($X^{\leq m}$, respectively).

Recall that for each r-edge colouring c of $K_\mathbb{N}$ Lemma 1.3.5 gives a partition of the vertices, which we will denote by $d_c : \mathbb{N} \to \{0, \ldots, r-1\}$, and a special colour $i_c < r$. We define a set $A_s \subset \mathbb{N}$ for each finite sequence $s \in \{0, \ldots, r-1\}^{\leq (k-1)r+1}$ by induction on $|s|$ as follows:

- let $A_\emptyset = \mathbb{N}$,
- if A_s is defined and finite then let

$$A_{s^\frown 0} = A_s \text{ and } A_{s^\frown i} = \emptyset \text{ for } 1 \leq i < r, \tag{2.10}$$

- if A_s is defined and infinite then let

$$A_{s^\frown i} = \{u \in A_s : d_{c\restriction A_s}(u) = i\} \text{ for } i < r. \tag{2.11}$$

Fix an arbitrary $s \in \{0, \dots, r-1\}^{(k-1)r+1}$ such that A_s is infinite. Then there is a colour $i_s < r$ and a k-element subset $H_s = \{h_1 > h_2 > \dots > h_k\}$ of $\{0, \dots, (k-1)r\}$ such that

$$s(h_j) = i_s$$

for all $j = 1, \dots, k$. Let $W_0 = A_s$ and $W_j = A_{s\restriction h_j}$ for $j = 1, \dots, k$. Note that the choice of i_s ensures that

$$W_{j+1} \cap N_{G_s}[F] \text{ is infinite}$$

for each $j < k$ and finite set $F \subset \bigcup_{i \leq j} W_i$, where $G_s = (\mathbb{N}, c^{-1}\{i_s\})$. Thus, by Theorem 2.3.5, Bob has a winning strategy in the game $\mathfrak{G}_k(G_s, A_s)$.

Playing the games

$$\{\mathfrak{G}_k(G_s, A_s) : s \in \{0, \dots, r-1\}^{(k-1)r+1} \text{ and } A_s \text{ is infinite}\} \tag{2.12}$$

simultaneously, that is, applying Lemma 2.3.4 we can find at most $r^{(k-1)r+1}$ many k^{th} powers of disjoint monochromatic paths which cover $\mathbb{N}$ apart from the finite set $\bigcup\{A_s : A_s \text{ is finite}\}$. □

In the case of $k = r = 2$, we have the following stronger result:

Theorem 2.3.7. *(1) Given an edge colouring of $K_\mathbb{N}$ with 2 colours, the vertices can be partitioned into ≤ 4 monochromatic path-squares (that is, second powers of paths):*

$$K_\mathbb{N} \sqsubset (\mathfrak{PathSquare})_{2,4}.$$

(2) The result above is sharp: there is an edge colouring of $K_\mathbb{N}$ with 2 colours such that the vertices cannot be covered by 3 monochromatic path-squares:

$$K_\mathbb{N} \not\sqsubset (\mathfrak{PathSquare})_{2,3}.$$

To prove Theorem 2.3.7 we need some further preparation. First, in [76, Corollary 1.10] Pokrovskiy proved the following: *Let $k, n \geq 1$ be natural numbers. Suppose that the edges of K_n are coloured with two colours. Then the vertices of K_n can be covered with k disjoint paths of colour 1 and a disjoint k^{th} power of a path of colour 0.*

Second, we will apply the following

Lemma 2.3.8. *Assume that $P = v_0, v_1, \dots$ is a finite or one-way infinite path in a graph G and there is $W \subset V(G) \setminus P$ so that*

$$(W \cap \mathrm{N}_G[\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}]) \text{ is infinite for all } v_i \in P. \tag{2.13}$$

Let $\mathcal{F}$ be a countable family of infinite subsets of W. Then G contains a square of a path R which covers P while $R \setminus P \subset W$, and $F \setminus R$ is infinite for all $F \in \mathcal{F}$. Moreover, if P is finite then R can also be *chosen to be finite.*

Proof. Let $F_0, F_1, \dots$ be an enumeration of $\mathcal{F}$ in which each element shows up infinitely often.

Pick distinct vertices $w_0, f_0, w_1, f_1, \dots$ from W such that

$$w_i \in \mathrm{N}_G[\{v_{2i}, v_{2i+1}, v_{2i+2}, v_{2i+3}\}] \text{ and } f_i \in F_i.$$

Then

$$R = v_0, v_1, w_0, v_2, v_3, w_1, v_4, \dots, v_{2i}, v_{2i+1}, w_i, v_{2i+2}, v_{2i+3}, w_{i+1}, \dots \tag{2.14}$$

is a square of a path which covers P, $R \setminus P \subset W$, and $\{f_n : n \in \mathbb{N}, F_n = F\} \subseteq F \setminus R$ for all $F \in \mathcal{F}$.

The last statement concerning the finiteness of R is obvious. $\square$

Proof of Theorem 2.3.7(1). Fix a colouring $c : [\mathbb{N}]^2 \to \{0,1\}$ and let $G_i = (\mathbb{N}, c^{-1}\{i\})$ for $i < 2$.

We will use the notation of Lemma 1.3.5. Let $c_0 = c$ and let

$$A_0 = \{v \in \mathbb{N} : d_{c_0}(v) = i_{c_0}\} \text{ and } B_0 = \mathbb{N} \setminus A_0. \tag{2.15}$$

Let $c_1 = c_0 \restriction B_0$ and provided B_0 is infinite we let

$$A_1 = \{v \in B_0 : d_{c_1}(v) = i_{c_1}\} \text{ and } B_1 = B_0 \setminus A_1. \tag{2.16}$$

We can assume that $i_{c_0} = 0$ without loss of generality.

Case 1: *B_0 is finite.*

First, $G[B_0]$ can be written as the disjoint union of two paths P_0 and P_1 of colour 1 and a square of a path Q of colour 0 by the above mentioned result of Pokrovskiy [76, Corollary 1.10]. Applying Lemma 2.3.8 for $G = G_1$, $P = P_0$, $W = A_0$ and $\mathcal{F} = \emptyset$ it follows that there is a finite square of a path R_0 in colour 1 which covers P_0 and $R_0 \setminus P_0 \subset A_0$. Applying Lemma 2.3.8 once more for $G = G_1$, $P = P_1$, $W = A_0 \setminus R_0$ and $\mathcal{F} = \emptyset$ it follows that there is a finite square of a path R_1 in colour 1 which covers P_1, and $R_1 \setminus P_1 \subset A_0 \setminus R_0$. Let $A_0' = A_0 \setminus (R_0 \cup R_1)$.

Now, by Theorem 2.3.5, Bob wins the game $\mathfrak{G}_2(G_0, A_0')$ witnessed by the sequence (A_0', A_0', A_0'); thus $G[A_0']$ can be covered by a single square of a path S of colour 0 by Lemma 2.3.4. That is, G can be covered by 4 disjoint monochromatic squares of paths: R_0, R_1, Q and S.

Case 2: *B_0 is infinite and $i_{c_1} = 0$.*

Note that, by Theorem 2.3.5, Bob wins the games

(i) $\mathfrak{G}_2(G_0, A_0)$ witnessed by (A_0, A_0, A_0),

(ii) $\mathfrak{G}_2(G_0, A_1)$ witnessed by (A_1, A_1, A_1),

(iii) $\mathfrak{G}_2(G_1, B_1)$ witnessed by (B_1, A_1, A_0).

Hence, the vertices can be partitioned into two squares of paths of colour 0 and a single square of a path of colour 1 by Lemma 2.3.4.

Case 3: *B_0 is infinite and $i_{c_1} = 1$.*

Since we applied Lemma 1.3.5 twice to obtain A_0 and B_0, and A_1 and B_1, and $B_1 \subseteq B_0$ we know that

(a) Bob wins the game $\mathfrak{G}_2(G_0, A_0)$ witnessed by (A_0, A_0, A_0);

(b) Bob wins the game $\mathfrak{G}_2(G_1, A_1)$ witnessed by (A_1, A_1, A_1);

(c) $N[F, 1] \cap A_0$ is infinite for every finite set $F \subset B_1$;

(d) $N[F, 0] \cap A_1$ is infinite for every finite set $F \subset B_1$;

(e) $N[F, 0] \cap A_0$ is infinite for every finite set $F \subset A_0$;

(f) $N[F, 1] \cap A_1$ is infinite for every finite set $F \subset A_1$.

First, partition B_1 into two paths P_0 and P_1 of colour 0 and 1, respectively. Indeed, if B_1 is infinite this can be done by Theorem 1.3.8 and if B_1 is finite one considers two disjoint paths P_0 and P_1 in B_1 of colour 0 and 1 with $|P_0| + |P_1|$ maximal (as outlined in a footnote in [29]); it is easily seen that $P_0 \cup P_1$ must be B_1.

Now, our plan is to cover P_0 and P_1 with disjoint squares of paths R_0 and R_1 of colour 0 and 1, respectively, such that $R_0 \setminus P_0 \subset A_1$, $R_1 \setminus P_1 \subset A_0$ while

(a') Bob wins the game $\mathfrak{G}_2(G_0, A_0 \setminus R_1)$ witnessed by $(A_0 \setminus R_1, A_0 \setminus R_1, A_0 \setminus R_1)$,

(b') Bob wins the game $\mathfrak{G}_2(G_1, A_1 \setminus R_0)$ witnessed by $(A_1 \setminus R_0, A_1 \setminus R_0, A_1 \setminus R_0)$.

Let

$$\mathcal{F}_0 = \{N[F, 0] \cap A_0 : F \subset A_0 \text{ finite}\},$$

and

$$\mathcal{F}_1 = \{N[F, 1] \cap A_1 : F \subset A_1 \text{ finite}\},$$

and note that these families consist of infinite sets by (e) and (f) above. Apply Lemma 2.3.8 for $G = G_0$, $W = A_1$, $P = P_0$ and $\mathcal{F} = \mathcal{F}_1$ to find a square of a path R_0 in G_0 which covers P_0, $R_0 \setminus P_0 \subset A_1$ and $F \setminus R_0$ is infinite for all $F \in \mathcal{F}_1$, that is,

$$N[F, 1] \cap (A_1 \setminus R_0) \text{ is infinite for every finite set } F \subset A_1. \tag{2.17}$$

Apply Lemma 2.3.8 once more for $G = G_1$, $W = A_0$, $P = P_1$ and $\mathcal{F} = \mathcal{F}_0$ to find a square of a path R_1 in G_1 with $R_1 \setminus P_1 \subset A_0$ which covers P_1 and $F \setminus R_1$ is infinite for all $F \in \mathcal{F}_0$, that is,

$$N[F, 0] \cap (A_0 \setminus R_1) \text{ is infinite for every finite set } F \subset A_0. \tag{2.18}$$

Then, by Theorem 2.3.5, (2.18) yields (a'), and (2.17) yields (b').

Hence $(A_0 \setminus R_1) \cup (A_1 \setminus R_0)$ can be partitioned into two monochromatic squares of paths by Lemma 2.3.4 which in turn gives a partition of all the vertices into 4 monochromatic squares of paths. □

Proof of Theorem 2.3.7(2). Fix a partition (A, B, C, D) of $\mathbb{N}$ such that A is infinite, $|B| = |C| = 4$, and $|D| = 1$. Define the colouring $c : [\mathbb{N}]^2 \to \{0, 1\}$ as follows see Figure 2.4:

$$c^{-1}\{1\} = \{\{a, v\} : a \in A, v \in B \cup C \cup D\} \cup [B]^2 \cup [C]^2. \tag{2.19}$$

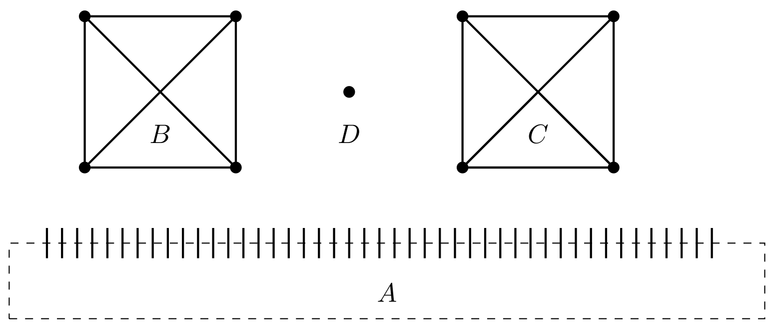


Figure 2.4: The example for Theorem 2.3.7(2)

If P is a monochromatic square of a path which intersects both A and $B \cup C \cup D$, then P should be in colour 1, so $P \cap A$ should be finite. Thus every partition of $K_{\mathbb{N}}$ into monochromatic squares of paths should contain an infinite 0-monochromatic square of a path $S \subset A$.

It suffices to show now that $B \cup C \cup D$ cannot be covered by two monochromatic squares of paths. Let $D = \{d\}$.

First, if P is a 1-monochromatic square of a path then $P' = P \cap (B \cup C \cup D)$ is a 1-monochromatic path. As two 1-monochromatic paths cannot cover $B \cup C \cup D$, two 1-monochromatic squares of paths will not cover $B \cup C \cup D$ neither.

Second, if Q is a 0-monochromatic square of a path which intersects $B \cup C \cup D$ then $Q \subset B \cup C \cup D$. Assume that $d \notin Q$ and let $Q = x_1, x_2, \dots$. If $x_1 \in B$ then $x_2 \in C$ so x_3 does not exists because Q is 0-monochromatic square of a path. Hence $d \notin Q$ implies $|Q \cap B| \leq 1$ and $|Q \cap C| \leq 1$. If $d \in Q$, then cutting Q into two by d and using the observation above we yield that $|Q \cap B| \leq 2$ and $|Q \cap C| \leq 2$. In turn, two 0-monochromatic squares of paths cannot cover $B \cup C \cup D$.

Finally using just one 0-monochromatic square of a path Q we cannot cover $(B \cup C) \setminus Q$ by a single 1-monochromatic square of a path because there is no 1-coloured edge between $B \setminus Q \neq \emptyset$ and $C \setminus Q \neq \emptyset$. □

2.4 Further results and open problems

In general, there are two directions in which one can aim to extend current results: investigate edge coloured non-complete graphs; determine the exact number of monochromatic structures (paths, powers of paths) needed to cover a certain edge coloured graph.

The next results shows Conjecture 2.1.9 for the graph $K_{\omega,\omega}$:

Theorem 2.4.1. *Let $c : E(K_{\omega,\omega}) \to r$ for some $r \in \mathbb{N}$. Then $K_{\omega,\omega}$ can be partitioned into at most $2r-1$ monochromatic paths. Furthermore, for every $r \in \mathbb{N}$ there is $c_r : E(K_{\omega,\omega}) \to r$ so that $K_{\omega,\omega}$ cannot be covered by less than $2r-1$ monochromatic paths.*

Proof. Let us denote the two classes of $K_{\omega,\omega}$ by A and B. Fix a colouring c and ultrafilters U_A, U_B on A, B respectively; now, let $A_i = \{u \in A : \{v \in B : c(u,v) \in i\} \in U_B\}$ and similarly $B_i = \{v \in B : \{u \in A : c(u,v) \in i\} \in U_A\}$. Without loss of generality, we can suppose that $A_0 \in U_A$. Let H_i denote the graph on $A \cup B$ with edges $c^{-1}(i)$.

Claim 2.4.2. *Bob wins the games $\mathfrak{G}_1(H_0, A_0 \cup B_0), \mathfrak{G}_1(H_i, A_i)$ and $\mathfrak{G}_1(H_i, B_i)$ for $1 \leq i < r$.*

Proof. It is easy to see that Claim 2.3.3 can be applied in each case. □

This finishes the proof of the first part of the theorem by Lemma 2.3.4.

Next, we will construct our colourings c showing that the above result is sharp. Let $r \geq 2$, let $A = \cup\{A_i : i < r\}$ with A_0 infinite and $A_i = \{a_i\}$ for $1 \leq i < r$ and let $B = \cup\{B_i : i < r\}$ with each B_i infinite. Define the r-colouring c_r as follows: let

$$c_r \restriction A_i \times B_j = i + j \mod r \quad \text{for } i, j \in r.$$

Note that if P is a monochromatic path which covers some A_i then $|\{j < r : P \cap B_j \neq \emptyset\}| \leq 1$; furthermore P is finite and thus $B_j \setminus P \neq \emptyset$ if $1 \leq i < r$ and $j < r$. Similarly, if P is a monochromatic path which covers some B_i then $|\{j < r : P \cap A_j \neq \emptyset\}| \leq 1$ as well. Now it is easy to see that there is no c_r-monochromatic cover by less than $2r - 1$ paths. □

We mention that Rado's original result (Theorem 2.1.2) implies

Claim 2.4.3. *For every r-edge colouring of the random graph on $\mathbb{N}$ we can partition the vertices into r disjoint paths of distinct colours.*

Regarding Theorem 2.3.6 we ask the following most general question:

Problem 2.4.4. *What is the exact number of monochromatic k^{th} powers of paths needed to partition the vertices of an r-edge coloured complete graph on $\mathbb{N}$?*

Naturally, any result aside from the resolved case of $k = r = 2$ (see Theorem 2.3.7) would be very welcome. In particular:

Problem 2.4.5. *Can we bound the number of monochromatic k^{th} powers of paths needed to partition the vertices of an r-edge coloured complete graph on $\mathbb{N}$ by a function of r and k?*

Chapter 3

Path decompositions of uncountable graphs

The goal of this chapter to prove that every finite edge coloured infinite complete graph can be partitioned into disjoint monochromatic paths of different colours. The smallest uncountable case of this theorem with two colours was proved jointly by M. Elekes, L. Soukup and Z. Szentmiklóssy [14] while the general result is due to the present author [93].

We are not aware of any results on uncountable paths in graphs prior to our work. We mention a paper by A. Hajnal, P. Komjáth, L. Soukup and I. Szalkai [44] where the authors considered the question if the vertices of a μ-edge coloured complete graph can be partitioned into τ pieces, each connected in some colour. Our monochromatic path decomposition gives a very special partition with the above property when $\mu = \tau$ is finite and we return to the results of [44] in Section 3.5.

3.1 Constructing uncountable paths

Now, we present our most important tools in constructing uncountable paths with Lemma 3.1.8 being the main result of this section.

Definition 3.1.1. *For a path P and $x <_P y \in P$ let $P \restriction [x, y)$ denote the segment of P from x to y (excluding y itself). For a set A and path P we say that P* is concentrated on *A iff*

$$N(y) \cap A \cap P \restriction [x, y) \neq \emptyset$$

for every $<_P$-limit $y \in P$ and $x <_P y$ in P.

We will use the following easy observation regularly

Observation 3.1.2. *Suppose that P is a path concentrated on a set A, $p \in V(P)$ and there is a limit element of P above p. Then there is a $q \in A \cap V(P)$ such that $p <_P q$ and $P \restriction [p, q)$ is finite.*

We will apply the next lemma multiple times:

Lemma 3.1.3. *Let $G = (V, E)$ be a graph and $A \in [V]^\kappa$ κ-unseparable. Then the following are equivalent:*

1. *there is a path P of order type κ concentrated on A,*

2. *A is covered by a path Q of order type κ concentrated on A.*

Moreover, if $a \in A$ and $C \in [A]^{cf(\kappa)}$ then we can construct Q with first point a and cofinal set C.

Proof. We prove by induction on κ; the result holds for $\kappa = \omega$ by Observation 1.3.4 so suppose that $\kappa > \omega$ and that we proved for cardinals $< \kappa$. Also, fix $a \in A$, $C \in [A]^{cf(\kappa)}$ and path P concentrated on A; note that we do not need to worry about C if κ is regular as every subset of A of size κ will be cofinal in Q. We distinguish two cases:

Case 1: $\kappa > cf(\kappa)$.

Let us fix an increasing cofinal sequence of regular cardinals $(\kappa_\alpha)_{\alpha<cf(\kappa)}$ in κ so that $\kappa_0 = cf(\kappa)$ and $\kappa_\beta > \sup\{\kappa_\alpha : \alpha < \beta\}$ for all $\beta < cf(\kappa)$.

Claim 3.1.4. *There are pairwise disjoint paths $\{R_\alpha : \alpha < cf(\kappa)\}$ in $V \setminus (\{a\} \cup C)$ concentrated on A such that*

1. *R_0 has order type κ_0, R_α has order type $\kappa_\alpha + n_\alpha$ for some $n_\alpha \in \omega \setminus \{0\}$,*

2. *R_0 starts in A, R_α starts and finishes in A for $0 < \alpha < cf(\kappa)$,*

3. *for every $x, y \in A$ there are κ many pairwise disjoint finite paths from x to y in $\{x, y\} \cup V \setminus \bigcup_{\alpha<cf(\kappa)} R_\alpha$.*

Proof. Let $A = \bigcup\{A_\alpha : \alpha < cf(\kappa)\}$ be an increasing union with $|A_\alpha| \leq \kappa_\alpha$. We proceed by induction on $\alpha < cf(\kappa)$ and construct $\{R_\alpha : \alpha < \beta\}$ satisfying 1. and 2. above and sets $\{W_\alpha : \alpha < \beta\}$ so that

(a) $W_\alpha \in [V]^{\kappa_\alpha}$ and any two points $x \neq y \in A_\alpha$ can be connected by κ_α pairwise disjoint finite paths in $\{x, y\} \cup W_\alpha$, and

(b) $W_\alpha \cap R_{\alpha'} = \emptyset$ for $\alpha, \alpha' < \beta$.

First choose R_0 to be a segment of P which satisfies the above conditions (on the starting point and order type) and choose $W_0 \subseteq V \setminus R_0$ using that A_0 is κ_0-unseparable.

In general, let $X_\beta = \bigcup\{R_\alpha : \alpha < \beta\} \cup \bigcup\{W_\alpha : \alpha < \beta\}$ and note that X_β has size less than κ. As the path P has κ many κ_β-limit points, we can select a subpath R_β of P (an interval of P starting and finishing in A) of order type $\kappa_\beta + n_\beta$ such that $R_\beta \cap X_\beta = \emptyset$. We can construct now $W_\beta \subset V \setminus (X_\beta \cup R_\beta)$ as desired using that A_β is κ-unseparable. $\square$

Let C_α denote a subset of $A \cap R_\alpha$ which is cofinal in $R_\alpha \restriction \kappa_\alpha$ and let t_α denote the κ_α-limit point of R_α for $0 < \alpha < cf(\kappa)$. Write $A \setminus \bigcup_{\alpha<cf(\kappa)} R_\alpha$ as $\{A_\alpha : \alpha < cf(\kappa)\}$ so that $|A_\alpha| \leq \kappa_\alpha$. List C as $\{c_\alpha : \alpha < cf(\kappa)\}$.

Construct a sequence of paths $\{Q_\alpha : \alpha < cf(\kappa)\}$ concentrated on A so that

1. Q_β end extends Q_α for $\alpha < \beta < cf(\kappa)$,

2. Q_α starts with a and finishes with a point $r_\alpha \in A \cap R_0$,

3. $Q_\alpha \cap R_0 \subset R_0 \restriction r_\alpha \cup \{r_\alpha\}$ and Q_α covers all points $x \in A \cap R_0$ such that $x <_{R_0} r_\alpha$,

4. $(Q_{\alpha+1} \setminus Q_\alpha) \cap C = \{c_\alpha\}$,

5. Q_α covers $A_\alpha \cup (R_\alpha \cap A)$ for $0 < \alpha < cf(\kappa)$,

6. $Q_\alpha \cap R_\beta = \emptyset$ if $\alpha < \beta < cf(\kappa)$.

Suppose we have Q_α for $\alpha < \beta$. If $\beta = \alpha + 1$ then let $r_\beta^- = r_\alpha$, if β is limit then let r_β^- be the first limit point of R_0 above $\{r_\alpha : \alpha < \beta\}$. Note that $r_\beta^- = \sup_{<_{R_0}}\{r_\alpha : \alpha < \beta\}$ if β is a limit and hence $Q_{<\beta} = \bigcup\{Q_\alpha : \alpha < \beta\} \cup \{r_\beta^-\}$ is a path concentrated on A by property (3). Let $r_\beta^+ <_{R_0} r_\beta \in R_0$ be the first two points of A above r_β^- and note that that $R_0 \restriction [r_\beta^-, r_\beta]$ is finite.

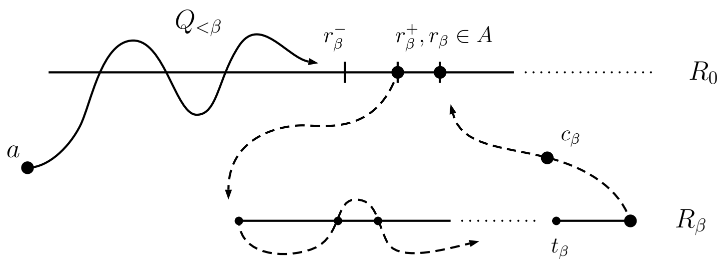


Figure 3.1: Extending $Q_{<\beta}$ to Q_β.

Claim 3.1.5. *There is a path S concentrated on A in $V \setminus (\bigcup\{R_\alpha : \alpha \in \kappa \setminus \{\beta\}\} \cup Q_{<\beta})$ such that*

(i) S end extends $R_0 \restriction [r_\beta^-, r_\beta^+]$ and ends in r_β,

(ii) S covers $A_\beta \setminus Q_{<\beta} \cup (R_\beta \cap A)$,

(iii) $S \cap C = \{c_\beta\}$.

Proof. S is constructed using R_β and the inductive hypothesis for κ_β. First, let us find a finite path S' starting with t_β and the finite end segment of R_β so that $S' \cap C = \{c_\beta\}$ and S' ends in r_β. This can be done as A is κ-unseparable.

Now, note that $R_\beta \restriction \kappa_\beta$ is a path of order type κ_β concentrated on $A_\beta \setminus Q_{<\beta} \cup (R_\beta \cap A)$ in $V_\beta = V \setminus (\bigcup\{R_\alpha : \alpha \in \kappa \setminus \{\beta\}\} \cup Q_{<\beta} \cup C \cup S')$ and that $A_\beta \setminus Q_{<\beta} \cup (R_\beta \cap A)$ is κ_β-unseparable in V_β. Hence, we can apply the inductive hypothesis in V_β and find a path S'' concentrated on A of order type κ_β which starts with r_β^+, covers $A_\beta \setminus Q_{<\beta} \cup (R_\beta \cap A)$ and has cofinal set C_β. We set $S = R_0 \restriction [r_\beta^-, r_\beta^+]^\frown S''^\frown S'$ □

Let $Q_\beta = Q_{<\beta}{}^\frown S$ which finishes the inductive step and hence the proof for the case when κ is singular.

Case 2: $\kappa = cf(\kappa)$.

We fix a nice sequence of elementary submodels $(M_\alpha)_{\alpha<\kappa}$ covering A with $A, G \in M_1$ and let $A_\alpha = M_\alpha \cap A$. Let $p_\alpha = \min_{<_P} P \setminus M_\alpha$ for $\alpha < \kappa$ and note that $p_\alpha \in M_{\alpha+1}$ and p_α is a $<_P$-limit. Also, observe that

$$\{p \in A \cap N(p_\beta) : p <_P p_\beta\} \setminus M_\alpha \text{ is infinite}$$

for all $\alpha < \beta < \kappa$; indeed, this follows from the fact that $M_\alpha \cap P$ is a proper initial segment of $P \restriction p_\beta$.

Now, it suffices to construct a sequence of paths $\{Q_\alpha : \alpha < \kappa\}$ concentrated on A so that Q_1 starts with a and

1. $A_\alpha \subset Q_\alpha \subset M_\alpha$,

2. $Q_\alpha{}^\frown(p_\alpha)$ is a path which is an initial segment of Q_β

for all $\alpha < \beta < \kappa$. Indeed, $\bigcup\{Q_\alpha : \alpha < \kappa\}$ is the path we are looking for.

Suppose we constructed Q_α for $\alpha < \beta$. Let

$$Q_{<\beta} = \begin{cases} \bigcup\{Q_\alpha : \alpha < \beta\}^\frown(p_\beta) & \text{if } \beta \text{ is a limit,} \\ Q_\alpha{}^\frown(p_\alpha) & \text{if } \beta = \alpha + 1. \end{cases}$$

Note that $Q_{<\beta}$ is a path; for successor β this is ensured by (2) while for a limit β ensured by (1) and the observation about p_β above.

If β is a limit, we simply let $Q_\beta = \bigcup\{Q_\alpha : \alpha < \beta\}$; it is easy to see that (1) is satisfied as the chain $(M_\alpha)_{\alpha<cf(\kappa)}$ is continuous.

Now suppose $\beta = \alpha+1$. Our goal is to apply the inductive hypothesis and find a path S concentrated on A in $V \cap (M_{\alpha+1} \setminus M_\alpha)$ so that

(i) S starts at p_α,

(ii) S covers $A \cap M_{\alpha+1} \setminus M_\alpha$, and

(iii) there is an infinite subset of $N(p_{\alpha+1}) \cap A \cap M_{\alpha+1} \setminus M_\alpha$ cofinal in S.

Indeed, $Q_\beta = Q_\alpha{}^\frown S$ will satisfy (1) and (2).

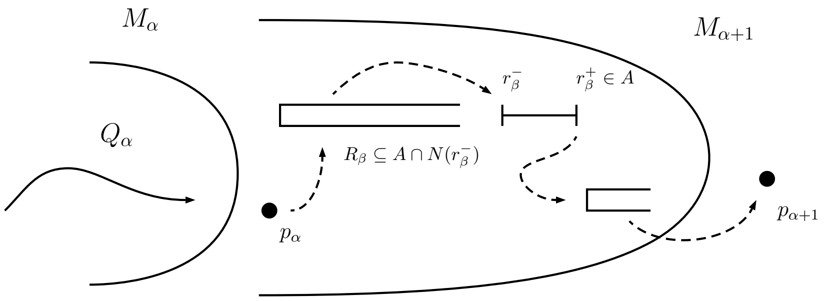


Figure 3.2: Extending $Q_{<\beta}$ to Q_β.

Let us pick the cofinal set mentioned in (iii) first: let $R^- \subseteq N(p_{\alpha+1}) \cap A \cap M_{\alpha+1} \setminus (M_\alpha \cup \{p_\alpha\})$ be infinite and find a path R of order type ω in $M_{\alpha+1} \setminus M_\alpha$ covering R^- and starting in R^-. The path S will end with R and hence property (iii) will be satisfied. Also, p_α might not be in A but a finite segment of P connects p_α to some $q \in A \cap M_{\alpha+1} \setminus M_\alpha$.

Now, let $\lambda = |A \cap M_{\alpha+1} \setminus M_\alpha|$ and find a point $r_\beta^- \in P \cap M_{\alpha+1} \setminus M_\alpha$ which is a $cf(\lambda)$-limit point of P; let r_β^+ be the first point of $A \cap P$ above r_β^-. Let $W = V(R \cup P \restriction [p_\alpha, q] \cup P \restriction [r_\beta^-, r_\beta^+])$.

Find a finite path T in $M_{\alpha+1} \setminus (M_\alpha \cup W)$ connecting r_β^+ to the first point of R and let $R' = P \restriction [r_\beta^-, r_\beta^+]^\frown T^\frown R$. The path S will start with $P \restriction [p_\alpha, q]$ and end with R'. Let $W' = W \cup V(T)$.

Finally, pick any $R_\beta \in [N(r_\beta^-) \cap A \cap M_{\alpha+1} \setminus (M_\alpha \cup W')]^{cf(\lambda)}$. Now apply the inductive hypothesis in the graph $G \restriction V \cap M_{\alpha+1} \setminus (M_\alpha \cup W')$ for the λ-unseparable set $A \cap M_{\alpha+1} \setminus (M_\alpha \cup W')$; we can find a path S' concentrated on A which starts with $P \restriction [p_\alpha, q]$, covers $A \cap M_{\alpha+1} \setminus M_\alpha$ and R_β is cofinal in S'. Note that $V \cap M_{\alpha+1} \setminus M_\alpha$ contains a path which is concentrated on $A \cap M_{\alpha+1} \setminus M_\alpha$ and has ordertype λ; indeed, consider an appropriate segment of the original path P in $M_{\alpha+1} \setminus M_\alpha$.

We are done by letting $S = S'^\frown R'$.

□

As we will see, there are three main ingredients to constructing a path covering a set A of size κ one of which is being κ-unseparable.

Definition 3.1.6. *Suppose that $G = (V, E)$ is graph and $A \subseteq V$. We say that A satisfies $\spadesuit_\kappa$ iff for all $X \in [V]^{<\kappa}$ and $\lambda < \kappa$ there is a path P of order type λ disjoint from X which is concentrated on A.*

If we have a fixed edge colouring we use $\spadesuit_{\kappa,i}$ for "$\spadesuit_\kappa$ in colour i" for short. Also, let us mention an easy result for later reference:

Observation 3.1.7. *Suppose that $G = (V, E)$ is a graph, $A \in [V]^\kappa$. Consider the following statements:*

1. *there is a path P in G of size κ concentrated on A,*
2. *A satisfies $\spadesuit_\kappa$,*
3. *for each $\lambda < \kappa$ there are κ-many pairwise disjoint paths concentrated on A of order type λ.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): suppose that $X \in [V]^{<\kappa}$ and $\lambda < \kappa$. If κ is regular then $X \cap P$ must be bounded in P and hence an end segment of P is a path disjoint from X which has order type κ. If κ is singular, then $\mu = |X|^+$ is less than κ and we repeat the previous argument for the $P \restriction \mu$.

(2)$\Rightarrow$(3): suppose that there is $\lambda < \kappa$ and a maximal family $\mathcal{P}$ of pairwise disjoint paths concentrated on A of order type λ so that $|\mathcal{P}| < \kappa$. Apply $\spadesuit_\kappa$ to $X = \cup\mathcal{P} \in [V]^{<\kappa}$ to extend $\mathcal{P}$. This contradicts the maximality of $\mathcal{P}$.

(3)$\Rightarrow$(2): suppose that $X \in [V]^{<\kappa}$ and $\lambda < \kappa$. Take a family $\mathcal{P}$ of pairwise disjoint paths concentrated on A of order type λ so that $|\mathcal{P}| = \kappa$. There is $P \in \mathcal{P}$ so that $P \cap X = \emptyset$.

□

Clearly, (2) does not imply (1) as $\spadesuit_\kappa$ is easily satisfied in a graph which has no connected component of size κ.

The next lemma will be our main tool in constructing paths.

Lemma 3.1.8. *Suppose that $G = (V, E)$ is a graph, $\kappa \geq \omega$, $A \in [V]^\kappa$ and*

1. *A is κ-unseparable, and*

if $\kappa > \omega$ then

2. *A satisfies $\spadesuit_\kappa$, and*
3. *there is a nice sequence of elementary submodels $(M_\alpha)_{\alpha<cf(\kappa)}$ covering A with $A, G \in M_1$ so that there is $x_\beta \in A \setminus M_\beta, y_\beta \in V \setminus M_\beta$ with $\{x_\beta, y_\beta\} \in E$ and*

$$|N_G(y_\beta) \cap A \cap M_\beta \setminus M_\alpha| \geq \omega$$

for all $\alpha < \beta < cf(\kappa)$.

Then A is covered by a path P concentrated on A.

Note that condition (3) only makes sense for $\kappa > \omega$; indeed, if A is countable and $A \in M_1$ then $A \subseteq M_1$ as well. However, every countably infinite ω-unseparable set A is covered by a path of order type ω.

Proof. If $\kappa = \omega$ then Observation 1.3.4 finishes the proof. Suppose $\kappa > \omega$.

Let us fix $(M_\alpha)_{\alpha < cf(\kappa)}$ and $\{x_\alpha, y_\alpha : \alpha < cf(\kappa)\}$ as above; we can suppose that $x_\alpha, y_\alpha \in M_{\alpha+1}$. Let $A_\alpha = M_\alpha \cap A$ for $\alpha < cf(\kappa)$. It suffices to construct a sequence of paths $\{P_\alpha : \alpha < cf(\kappa)\}$ concentrated on A so that

(i) P_β end extends P_α,

(ii) $A_\alpha \subseteq P_\alpha \subseteq M_\alpha$,

(iii) $N_G(y_{\alpha+1}) \cap A \cap M_{\alpha+1} \setminus M_\alpha$ is cofinal in $P_{\alpha+1}$

for all $\alpha < \beta < cf(\kappa)$. We set $P = \bigcup\{P_\alpha : \alpha < cf(\kappa)\}$ which finishes the proof.

Suppose we constructed P_α for $\alpha < \beta$ as above; if β is a limit ordinal then we set $P_\beta = \bigcup\{P_\alpha : \alpha < \beta\}$. Suppose that $\beta = \alpha + 1$; note that $P_\alpha{}^\frown(y_\alpha, x_\alpha)$ is still a path regardless whether α is a limit or successor by (ii) and (iii). It suffices to find a path $S \subset M_{\alpha+1} \setminus M_\alpha$ concentrated on A starting at x_α so that $N(y_{\alpha+1}) \cap A \cap M_{\alpha+1} \setminus M_\alpha$ is cofinal in S and $A_{\alpha+1} \setminus A_\alpha \subset S$; indeed, we set $P_{\alpha+1} = P_\alpha{}^\frown(y_\alpha)^\frown S$ which finishes the proof.

We will essentially repeat the proof of Lemma 3.1.3 in the regular case. Recall that $N(y_{\alpha+1}) \cap A \cap M_{\alpha+1} \setminus M_\alpha$ is infinite. First, find a path R of order type ω in $M_{\alpha+1} \setminus (M_\alpha \cup \{x_\alpha, y_\alpha\})$ so that

$$|R \cap N_G(y_{\alpha+1}) \cap A \cap M_{\alpha+1} \setminus M_\alpha| \geq \omega$$

and R starts at a point r so that $|N_G(r) \cap A_{\alpha+1} \setminus A_\alpha| \geq cf(\nu)$ where $\nu = |A_{\alpha+1} \setminus A_\alpha|$. The only difficulty here is to find such an r; if κ is limit we can use $\spadesuit_\kappa$ to find a path $Q \in M_{\alpha+1}$ concentrated on A of size $|M_{\alpha+1}|^+$ and r can be chosen to be the first $|cf(\nu)|$-limit of Q (note that $\nu < |M_{\alpha+1}|^+$). A finite segment of Q connects r to some $r' \in Q \cap A$ and we continue to construct R from this finite path. If κ is a successor then we must have $\kappa = \nu^+$ (by the definition of a nice sequence of models) and note that $|N_G(y_{\alpha+\nu}) \cap A \setminus A_\alpha| \geq cf(\nu)$ for any $\alpha < \kappa$. Reflecting this property into $M_{\alpha+1}$ we find $y, x \in M_{\alpha+1} \setminus (M_\alpha \cup \{x_\alpha, y_\alpha\})$ so that $\{y, x\} \in E$, $x \in A$ and $|N(y) \cap A_{\alpha+1} \setminus A_\alpha| \geq cf(\nu)$. We can start R by y and x and connect the rest of the points using that A is κ-unseparable in $M_{\alpha+1} \setminus M_\alpha$.

Now, we construct S with the above required properties so that it has order type $\nu + \omega$ and R is the last ω many points of S. Indeed, $\spadesuit_\kappa$ implies that $G \restriction (V \cap M_{\alpha+1} \setminus (M_\alpha \cup R \cup \{y_\alpha\}))$ contains a path of order type ν concentrated on $A_{\alpha+1} \setminus A_\alpha$ so by applying Lemma 3.1.3 we can find a path S' starting at x_α, concentrated on A and of order type ν which covers $A_{\alpha+1} \setminus (A_\alpha \cup R)$ while $N_G(r) \cap A$ is cofinal in S'; we set $S = S'{}^\frown R$ which finishes the proof.

$\square$

3.2 The existence of monochromatic paths

Our goal in this section is to find large monochromatic paths in certain edge coloured graphs G by finding a set $A \subseteq V(G)$ which satisfies all three conditions of Lemma 3.1.8.

3.2.1 Preparations

As we stated in the introduction, we aim to deal with certain non-complete graphs:

Definition 3.2.1. *We call a graph $G = (V, E)$ κ-*complete *iff $|V| \geq \kappa$ and*

$$|V \setminus N_G(x)| < \kappa$$

for all $x \in V$.

Let us start with some basic observations.

Observation 3.2.2. *1. Any κ-complete graph $G = (V, E)$ is $|V|$-complete.*

2. If $G = (V, E)$ is κ-complete then any subset $X \in [V]^\kappa$ spans a κ-complete subgraph.

Proof. (1) If G is κ-complete then $|V| \geq \kappa$ and hence $|V \setminus N_G(x)| < \kappa \leq |V|$ for all $x \in V$. Thus G is $|V|$-complete.

(2) If $X \subseteq V$ has size κ then $|X \setminus N_G(x)| \leq |V \setminus N_G(x)| < \kappa$. □

In order to carry out our proofs we need to introduce a class of graphs closely related to $H_{\kappa,\kappa}$.

Definition 3.2.3. *We say that a graph $G = (V, E)$ is of* type $H_{\kappa,\kappa}$ *iff $V = A \cup B$ where $A = \{a_\xi : \xi < \kappa\}, B = \{b_\xi : \xi < \kappa\}$ are 1-1 enumerations and*

$$\{a, b\} \in E(G) \text{ if } a = a_\xi, b = b_\zeta \text{ for some } \xi \leq \zeta < \kappa.$$

We will call A the *main class* of G and (A, B) with the inherited ordering is the $H_{\kappa,\kappa}$-decomposition of G. As before, we use the notation $G \restriction \lambda$ to denote $G \restriction \{a_\xi, b_\xi : \xi < \lambda\}$.

We will mainly apply this definition in two cases: when the classes A and B of the graph G of type $H_{\kappa,\kappa}$ are disjoint (i.e. G is isomorphic to *the graph $H_{\kappa,\kappa}$*) and when the main class equals $V(G)$.

Observation 3.2.4. *Suppose that $G = (V, E)$ is a graph, $A \in [V]^\kappa$ for some cardinal κ and A is the increasing union of sets $\{A_\alpha : \alpha < cf(\kappa)\}$ where $|A_\alpha| < \kappa$ and*

$$|N[A_\alpha]| = \kappa$$

for all $\alpha < cf(\kappa)$. Then there is a subgraph H of G of type $H_{\kappa,\kappa}$ with main class A.

Proof. Find an enumeration $A = \{a_\xi : \xi < \kappa\}$ so that for every $\zeta < \kappa$ there is $\alpha_\zeta < cf(\kappa)$ with $\{a_\xi : \xi \leq \zeta\} \subseteq A_{\alpha_\zeta}$. Hence

$$|N[\{a_\xi : \xi \leq \zeta\}]| \geq |N[A_{\alpha_\zeta}]| = \kappa.$$

Now, we can inductively find vertices $b_\zeta \in N[\{a_\xi : \xi \leq \zeta\}] \setminus \{b_\xi : \xi < \zeta\}$ for all $\zeta < \kappa$ and hence $A \cup \{b_\xi : \xi < \kappa\}$ is the type $H_{\kappa,\kappa}$ subgraph. □

Observation 3.2.5. *Suppose that $G = (V, E)$ is of type $H_{\kappa,\kappa}$ with main class V. Then there is a κ-complete graph embedded in G.*

Conversely, if $G = (V, E)$ is a κ-complete graph of size κ then G is of type $H_{\kappa,\kappa}$ with main class V.

Proof. If (A, B) is the $H_{\kappa,\kappa}$ decomposition of G then we have $B \subseteq A = V$ and $G \restriction B$ is the κ-complete subgraph.

Second, suppose that G is κ-complete and list V in type κ as $A = \{a_\xi : \xi < \kappa\}$. If κ is regular then $N[\{a_\xi : \xi < \zeta\}]$ has size κ for all $\zeta < \kappa$ hence Observation 3.2.4 finishes the proof.

If κ is singular then let us take an increasing, continuous and cofinal sequence $\{\kappa_\alpha : \alpha < cf(\kappa)\}$ in κ and let $A_\alpha = \{a_\xi : \xi < \kappa_\alpha, |V \setminus N(a_\xi)| < \kappa_\alpha\} \in [V]^{<\kappa}$. Note that $V = \bigcup_{\alpha < cf(\kappa)} A_\alpha$ is an increasing union of sets of size $< \kappa$ and $N[A_\alpha]$ has size κ. Again, Observation 3.2.4 can be applied to this new enumeration which finishes the proof. $\square$

Observation 3.2.6. *Suppose that H is of type $H_{\kappa,\kappa}$ for some cardinal κ. Then there is a path of order type κ which covers and is concentrated on the main class of H.*

This result is trivial for the graph $H_{\kappa,\kappa}$, however we have to be somewhat cautious when the two classes of H intersect.

Proof. Let $A = \{a_\xi : \xi < \kappa\}, B = \{b_\xi : \xi < \kappa\}$ witness that H is of type $H_{\kappa,\kappa}$. We define an increasing sequence of paths $\{P_\alpha : \alpha \in D\}$ where $D = \{\alpha < \kappa : \alpha \text{ is a limit of limits}\}$ by induction on α such that

1. P_α is a path concentrated on A,
2. $P_\alpha \cap A$ is a cofinal subset of P_α,
3. $P_\alpha \subseteq H \restriction \alpha + \omega + \omega$, and
4. P_α covers a_α

for all $\alpha \in D$.

Let us define P_0 inductively as $(p^0_n : n \in \omega)$ where

$$p^0_n = \begin{cases} a_{l_n} & \text{where } l_n = \min\{l \in \omega : a_l \notin \{p^0_m : m < n\}\} \text{ if } n \text{ is even,} \\ b_{\omega+k_n} & \text{where } k_n = \min\{k \in \omega : b_{\omega+k} \notin \{p^0_m : m < n\}\} \text{ if } n \text{ is odd.} \end{cases}$$

Suppose that P_α is defined for $\alpha < \beta$ where $\beta \in D$ and let $P_{<\beta} = \bigcup\{P_\alpha : \alpha \in \beta \cap D\}$. Let

$$\delta = \min\{\zeta \in \kappa : (P_{<\beta} \cup \{a_\beta\}) \subseteq H \restriction \zeta\}.$$

It is easy to see that $\delta \leq \beta$. Observe that $P_{<\beta}{}^\frown(b_{\delta+\omega})$ is a path concentrated on A. Let

$$P^-_\beta = \begin{cases} P_{<\beta}{}^\frown(b_{\delta+\omega}) & \text{if } a_\beta \in P_{<\beta}{}^\frown(b_{\delta+\omega}) \\ P_{<\beta}{}^\frown(b_{\delta+\omega}, a_\beta, b_{\delta+\omega+1}) & \text{if } a_\beta \notin P_{<\beta}{}^\frown(b_{\delta+\omega}). \end{cases}$$

By induction on $n < \omega$, define

$$p^\beta_n = \begin{cases} a_{\delta+l_n} & \text{where } l_n = \min\{l \in \omega : a_{\delta+l} \notin P^-_\beta \cup \{p^\beta_m : m < n\}\} \text{ if } n \text{ is even,} \\ b_{\delta+\omega+k_n} & \text{where } k_n = \min\{k \in \omega : b_{\delta+\omega+k} \notin P^-_\beta \cup \{p^\beta_m : m < n\}\} \text{ if } n \text{ is odd.} \end{cases}$$

We let

$$P_\beta = P^-_\beta{}^\frown(p^\beta_n)_{n\in\omega}.$$

Finally, set $P = \bigcup\{P_\alpha : \alpha \in D\}$ and note that P is a path concentrated on A which also covers A.

$\square$

Let $(IH)_{\kappa,r}$ denote the statement that

for any r-edge colouring of a graph G of type $H_{\kappa,\kappa}$ with main class A, there is an $X \subseteq A$ of size κ and $i < r$ so that X satisfies all three conditions of Lemma 3.1.8 in colour i.

Let $(IH)_\kappa$ denote

$(IH)_{\kappa,r}$ holds for all $r \in \omega$.

Note that in Lemma 1.3.5 we showed that $(IH)_\omega$ holds. Furthermore:

Observation 3.2.7. *For any G of type $H_{\kappa,\kappa}$, the main class of G satisfies all three conditions of Lemma 3.1.8. In particular, $(IH)_{\kappa,1}$ holds for all κ.*

Proof. Fix G of type $H_{\kappa,\kappa}$ with main class $A = \{a_\xi : \xi < \kappa\}$ and second class $B = \{b_\xi : \xi < \kappa\}$; we suppose $\kappa > \omega$. A is clearly κ-unseparable and $\spadesuit_\kappa$ is satisfied by Observation 3.2.6 and Observation 3.1.7. Now, for the third property take any nice sequence of elementary submodels $(M_\alpha)_{\alpha<cf(\kappa)}$ covering A with $A, G \in M_1$ and suppose that the enumeration $\{a_\xi : \xi < \kappa\}$ is also in M_1. Let $x_\beta = a_{\xi_\beta}, y_\beta = b_{\xi_\beta}$ where $\xi_\beta = \min(\kappa \setminus M_\beta)$.

Now $\{x_\beta, y_\beta\} \in E$, $x_\beta, y_\beta \in M_{\beta+1} \setminus M_\beta$ and we need to show that

$$|N(y_\beta) \cap A \cap M_\beta \setminus M_\alpha| \geq \omega$$

for all $\alpha < \beta$. Fix $\alpha < \beta$ and look at $\xi_\alpha = \min(\kappa \setminus M_\alpha)$. As $\xi_\alpha < \xi_\alpha + \omega < \xi_\beta$, we get that

$$\{a_{\xi_\alpha+i} : i < \omega\} \subseteq N(y_\beta) \cap A \cap M_\beta \setminus M_\alpha.$$

$\square$

From now on in this section, we work towards showing that $(IH)_\kappa$ holds for all κ.

3.2.2 The first main step

We wish to determine if a subset A of an edge coloured graph satisfies condition (3) of Lemma 3.1.8 in a given colour.

Lemma 3.2.8. *Let κ be an uncountable cardinal. Suppose that c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B). If $i < r$ then either*

(a) A satisfies condition (3) of Lemma 3.1.8 in colour i, or

(b) there is $\tilde{A} \in [A]^{<\kappa}$ so that $A \setminus \tilde{A}$ is covered by a graph H of type $H_{\kappa,\kappa}$ with main class $A \setminus \tilde{A}$ so that $i \notin \operatorname{ran}(c \restriction E(H))$.

We need the following claims:

Claim 3.2.9. *Suppose that $\kappa \geq cf(\kappa) = \mu > \omega$, c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B). Suppose that $\{M_\alpha : \alpha < \mu\}$ is a nice κ-chain of elementary submodels covering V with $G, A, B, c \in M_1$. If $i < r$ then either*

(a) there is a club $C \subseteq \mu$ so that for every $\beta \in C$ there is $x_\beta \in A \setminus M_\beta, y_\beta \in B \setminus M_\beta$ such that $c(x_\beta, y_\beta) = i$ and

$$|N(y_\beta, i) \cap A \cap M_\beta \setminus M_\alpha| \geq \omega$$

for all $\alpha < \beta$, or

(b) there is $\tilde{A} \in [A]^{<\kappa}$ so that $A \setminus \tilde{A}$ is covered by a graph H of type $H_{\kappa,\kappa}$ with main class $A \setminus \tilde{A}$ so that $i \notin \operatorname{ran}(c \restriction E(H))$.

Proof. Suppose that (a) fails i.e. there is a stationary set $S \subset \mu$ so that for all $\beta \in S$ and $x \in A \setminus M_\beta, y \in B \setminus M_\beta$ with $c(x, y) = i$ we have

$$|N(y, i) \cap A \cap M_\beta \setminus M_\alpha| < \omega$$

for some $\alpha < \beta$.

Let $G_{\neq i} = (V, c^{-1}(r \setminus \{i\}))$. Note that

Observation 3.2.10. *If there is an $\alpha \in S$ and $\lambda < \kappa$ so that*

$$|B \cap N(x, i)| \leq \lambda$$

for every $x \in A \setminus M_\alpha$ then (b) holds with $\tilde{A} = A \cap M_\alpha$.

Indeed, we can apply Observation 3.2.4 to $A \setminus \tilde{A}$ in the graph $G_{\neq i}$.

Otherwise, we distinguish two cases:

Case 1: κ is regular. Recall that we have $M_\alpha \cap \kappa \in \kappa$ and hence $x \in A \cap M_\alpha, y \in B \setminus M_\alpha$ implies $\{x, y\} \in E$.

Select $x_\beta \in A \setminus M_\beta$ and $y_\beta \in B \setminus M_\beta$ with $c(x_\beta, y_\beta) = i$; this can be done by Observation 3.2.10. Hence

$$|N(y_\beta, i) \cap A \cap M_\beta \setminus M_\alpha| < \omega$$

for some $\alpha < \beta$. That is, there is $\alpha(\beta) < \beta$ so that

$$N(y_\beta, i) \cap A \cap M_\beta \subset M_{\alpha(\beta)}$$

for all $\beta \in S \cap \lim(\omega_1)$ (where $\lim(\omega_1)$ denotes the set of limit ordinals in $\lim(\omega_1)$).

Apply Fodor's pressing down lemma to the regressive function $\beta \mapsto \alpha(\beta)$ on the stationary set $S \cap \lim(\omega_1)$ and find stationary $T \subseteq S \cap \lim(\omega_1)$ and $\tilde{\alpha} \in \kappa$ so that $\alpha(\beta) = \tilde{\alpha}$ for all $\beta \in T$. It is easy to see that (b) is satisfied with $\tilde{A} = A \cap M_{\tilde{\alpha}}$. Indeed, if $x \in A_\alpha = A \cap M_\alpha \setminus \tilde{A}$ and $\beta \in T \setminus \alpha$ then $\{x, y_\beta\} \in E$ and $c(x, y_\beta) \neq i$ (for any $\alpha \in \kappa \setminus \tilde{\alpha}$). In turn

$$|N_{G_{\neq i}}[A_\alpha]| \geq |\{y_\beta : \beta \in T \setminus \alpha\}| \geq \kappa.$$

Hence we can apply Observation 3.2.4 to $A \setminus \tilde{A}$ in $G_{\neq i}$.

Case 2: κ is singular. Recall that $\kappa_\alpha = |M_\alpha|$ is a strictly increasing cofinal sequence of cardinals in $\kappa \setminus cf(\kappa)$. Select $x_\beta \in A \setminus M_\beta$ and find $Y_\beta \in [B \setminus M_\beta]^{\kappa_\beta^+}$ so that $Y_\beta \subseteq N(x_\beta, i)$ for each $\beta \in S$; this can be done by Observation 3.2.10. We can suppose, by shrinking Y_β, that there is a finite set F_β and $\alpha(\beta) < \beta$ with

$$F_\beta = N(y,i) \cap A \cap M_\beta \setminus M_{\alpha(\beta)}$$

for all $y \in Y_\beta$. The importance here is that $\alpha(\beta)$ and $N(y,i) \cap A \cap M_\beta \setminus M_{\alpha(\beta)}$ does not depend on $y \in Y_\beta$ which can be done as there are only $|\beta|$ choices for $\alpha(\beta)$ and κ_β choices for F_β while κ_β^+ choices for $y \in Y_\beta$.

Apply Fodor's pressing down lemma to the regressive function $\beta \mapsto \alpha(\beta)$ and find a stationary $T \subseteq S$ and $\tilde{\alpha} \in cf(\kappa)$ so that $\alpha(\beta) = \tilde{\alpha}$ for all $\beta \in T$. Let $\tilde{A} = (A \cap M_{\tilde{\alpha}}) \cup \bigcup\{F_\beta : \beta \in T\}$ and note that $|\tilde{A}| < \kappa$.

As before, if $x \in A_\alpha = A \cap M_\alpha \setminus \tilde{A}$ and $\beta \in T \setminus \alpha$ then $\{x, y\} \in E$ and $c(x,y) \neq i$ for any $y \in Y_\beta$ and $\alpha \in \kappa \setminus \tilde{\alpha}$. In turn

$$|N_{G_{\neq i}}[A_\alpha]| \geq |\bigcup\{Y_\beta : \beta \in T \setminus \alpha\}| \geq \kappa.$$

Hence we can apply Observation 3.2.4 to $A \setminus \tilde{A}$ in $G_{\neq i}$.

□

Claim 3.2.11. *Suppose that $\kappa > \omega = cf(\kappa)$ and c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B). Suppose that $\{M_n : n \in \omega\}$ is a nice κ-chain of elementary submodels covering V with $G, A, B, c \in M_1$. If $i < r$ then either*

(a) there is an increasing sequence $\{n_k : k \in \omega\} \subseteq \omega$ with $n_0 = 0$ such that for all $k < \omega$ there is $x_k \in A \setminus M_{n_{k+1}}, y_k \in B \setminus M_{n_{k+1}}$ with $c(x_k, y_k) = i$ and

$$|N(y_k, i) \cap A \cap M_{n_{k+1}} \setminus M_{n_k}| \geq \omega,$$

or

(b) there is $\tilde{A} \in [A]^{<\kappa}$ so that $A \setminus \tilde{A}$ is covered by a graph H of type $H_{\kappa,\kappa}$ with main class $A \setminus \tilde{A}$ so that $i \notin \operatorname{ran}(c \restriction E(H))$.

Proof. Suppose that (a) fails; hence there is an $\ell \in \omega$ such that for every $x \in A \setminus M_n, y \in B \setminus M_n$ with $c(x,y) = i$ we have

$$|N(y,i) \cap A \cap M_n \setminus M_l| < \omega$$

for all $n \in \omega \setminus \ell$.

Observation 3.2.12. *If there is $n \in \omega$ and $\lambda < \kappa$ so that $N(x,i) \leq \lambda$ for all $x \in A \setminus M_n$ then (b) holds with $\tilde{A} = A \cap M_n$.*

Indeed, we can apply Observation 3.2.4 to $A \setminus \tilde{A}$ in the graph $G_{\neq i} = (V, c^{-1}(r \setminus \{i\}))$.

Otherwise, we can select $x \in A \setminus M_n$ and $Y_n \in [M_n \setminus M_\ell]^{|M_n|^+}$ with $Y_n \subset N(x_n, i)$ for all $n \in \omega \setminus \ell$. We can suppose, by shrinking Y_n, that there is a finite $a_n \subset A \cap M_n$ so that

$$N(y,i) \cap A \cap M_n \setminus M_\ell = a_n$$

for all $y \in Y_n$. Let $\tilde{A} = (A \cap M_\ell) \cup \bigcup\{a_n : n \in \omega \setminus \ell\}$. As before, in Case 2 of the proof of Claim 3.2.9, applying Observation 3.2.4 to $A \setminus \tilde{A}$ in $G_{\neq i}$ finishes the proof.

□

Hence, we arrived at the

Proof of Lemma 3.2.8. Assume that (b) fails in Lemma 3.2.8. Hence condition (b) of Claim 3.2.11 (if $cf(\kappa) = \omega$) or 3.2.9 (if $cf(\kappa) > \omega$) fails for colour i. In turn, we have a nice sequence of elementary submodels satisfying condition (3) of Lemma 3.1.8 in colour i by condition (a) of Claim 3.2.11 or 3.2.9. □

3.2.3 The second main step

Now, we would like to determine if, in an edge coloured graph of type $H_{\kappa,\kappa}$, a κ-unseparable subset satisfies $\spadesuit_\kappa$ in some colour.

Lemma 3.2.13. *Let κ be an infinite cardinal. Suppose that c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B). Let $I \in [r]^{<r}$, $X \in [A]^\kappa$ and suppose that X is κ-unseparable in all colours $i \in I$. If $(IH)_\lambda$ holds for $\lambda < \kappa$ then either*

(a) there is an $i \in I$ such that X satisfies $\spadesuit_{\kappa,i}$, or

(b) there is $\tilde{X} \in [X]^{<\kappa}$ and a partition $\{X_j : j \in r \setminus I\}$ of $X \setminus \tilde{X}$ such that

$$|N(x, j) \cap N(x', j) \cap B| = \kappa$$

for all $x, x' \in X_j$ and $j \in r \setminus I$.

In particular, the sets X_j given by condition (b) are κ-unseparable in colour j in $X_j \cup B$.

Moreover, if $B \subset X$ then there is $j \in I \setminus r$ so that X_j is κ-connected in colour j.

The proof of Lemma 3.2.13 (at the end of Section 3.2.3) will be achieved through a series of claims below. The main application of Lemma 3.2.13 is in the proof of Theorem 3.2.20.

Definition 3.2.14. *Suppose that λ is a cardinal, $G = (V, E)$ is graph with an r-edge colouring c. A* λ-configuration *in colours $I \subseteq r$ is a pairwise disjoint family $\mathcal{X} = \{a_\xi : \xi < \lambda\} \subset [V]^{<\omega}$ and points $\mathcal{Y} = \{y_\xi : \xi < \lambda\}$ such that*

$$y_\zeta \in \bigcup\{N(x, i) : x \in a_\xi, i \in I\}$$

for all $\xi \le \zeta < \lambda$.

Claim 3.2.15. *Suppose that λ is a cardinal, $G = (V, E)$ is graph with an r-edge colouring c. Let $\mathcal{X}, \mathcal{Y}$ be a λ-configuration in colours $I \subseteq r$. Suppose that for each $i \in I$ there is $Y_i \subseteq V$ so that $\bigcup \mathcal{X}$ is λ-unseparable in colour i inside $V_i = \bigcup \mathcal{X} \cup Y_i$.*

Then $(IH)_{\lambda,|I|}$ implies that there is an $i \in I$ and a path P in colour i concentrated on $\bigcup \mathcal{X}$ which is inside V_i and has order type λ.

Proof. Let $\mathcal{X} = \{a_\xi : \xi < \lambda\}$ and $\mathcal{Y} = \{y_\xi : \xi < \lambda\}$ denote the λ-configuration. By setting $a'_\xi = \bigcup\{a_{\xi+i} : i < |I| + 1\}$ and $y'_\xi = y_{\xi+|I|+1}$ for $\xi < \lambda$ limit we get that for all limit ordinals $\xi \le \zeta < \lambda$ there is an $i \in I$ so that

$$|\{x \in a'_\xi : c(x, y'_\zeta) = i\}| \ge 2.$$

As $\{a'_\xi : \xi < \lambda \text{ limit}\}, \{y'_\xi : \xi < \lambda \text{ limit}\}$ is also a λ-configuration in colours I, we will suppose that the original λ-configuration had this property already.

Also, by thinning out, we can suppose that $\bigcup \mathcal{X} \cap \mathcal{Y} = \emptyset$ and for all $i \in I$, $\xi < \lambda$ and $x, x' \in a_\xi$ there are λ many disjoint finite i-monochromatic paths in V_i from x to x' which avoid $\mathcal{Y}$ and all other points of $\bigcup \mathcal{X}$.

Define a colouring of the graph $H_{\lambda,\lambda}$ by

$$d((\xi, 0), (\zeta, 1)) = i \text{ iff } |\{x \in a_\xi : c(x, y_\zeta) = i\}| \geq 2$$

and i is minimal such. Note that d is well defined by our previous preparation. Now $(IH)_{\lambda,|I|}$ implies that there is a path Q of colour i and size λ concentrated on the main class of $H_{\lambda,\lambda}$ for some $i \in I$.

Subclaim 3.2.16. *There is a path P of colour i and order type λ in $G \upharpoonright V_i$ concentrated on $\bigcup \mathcal{X}$.*

Proof. Let $Q = \{q_\nu : \nu < \lambda\}$ witness the path ordering; recall that each point q_ν in Q corresponds to a finite set $a_{\xi(\nu)}$ or a single vertex $\{y_{\xi(\nu)}\}$ from the λ-configuration and we identify q_ν with this set. Moreover, q_ν must be of the form $y_{\xi(\nu)}$ for every limit $\nu < \lambda$ as Q is concentrated on the main class of $H_{\lambda,\lambda}$.

Our goal is to define disjoint finite paths R_ν of colour i in $G \upharpoonright V_i$ so that $q_\nu \subset R_\nu$ while the concatenation $(R_\nu : \nu < \lambda)$ gives a path of colour i in $G \upharpoonright V_i$.

Construct $(R_\nu : \nu < \lambda)$ by induction on $\nu < \lambda$ so that

(i) R_ν is a finite path of colour i in $G \upharpoonright V_i$ and $R_\nu \cap (\bigcup \mathcal{X} \cup \mathcal{Y}) = q_\nu$,

(ii) $R_\nu \cap R_\mu = \emptyset$ if $\nu < \mu < \lambda$,

(iii) $R_\nu = q_\nu$ if $q_\nu = \{y_{\xi(\nu)}\}$,

moreover, if $q_\nu = a_{\xi(\nu)}$ then $\nu = \mu + 1$ and we make sure that

(iv) the first point of R_ν is a vertex $v \in a_{\xi(\nu)}$ so that $c(v, y_{\xi(\mu)}) = i$, and

(v) the last point of R_ν is a vertex $w \in a_{\xi(\nu)}$ so that $c(w, y_{\xi(\nu+1)}) = i$.

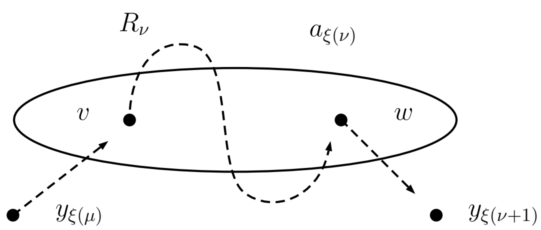


Figure 3.3: Constructing R_ν.

If we can achieve this, $(R_\nu : \nu < \lambda)$ gives a path of colour i concentrated on A.

Note that the only difficulty in this construction is to satisfy the last two requirements; indeed, we always have λ many disjoint finite paths of colour i connecting two arbitrary points of any a_ξ (avoiding all other points in question).

How to find the first and last point of R_ν if $q_\nu = a_{\xi(\nu)}$? As $\nu = \mu + 1$ for some $\mu < \lambda$ and by the definition of a path and the colouring d on $H_{\lambda,\lambda}$ we have

$$c(v, y_{\xi(\mu)}) = i \text{ for some } v \in a_{\xi(\nu)}$$

and we pick a single such $v \in a_{\xi(\nu)}$ which in turn satisfies (iv) above.

Second, $d(q_\nu, q_{\nu+1}) = i$ hence $\{x \in a_{\xi(\nu)} : c(x, y_{\xi(\nu+1)}) = i\}$ has at least two elements so we can pick

$$w \in \{x \in a_{\xi(\nu)} : c(x, y_{\xi(\nu+1)}) = i\} \setminus \{v\}$$

which will satisfy (v) above.

□

□

Claim 3.2.17. *Let κ be an infinite cardinal and $\lambda \leq cf(\kappa)$. Suppose that c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B) and let $I \subset r$. If for every $\tilde{A} \in [A]^{<\lambda}$ there is $a \in [A \setminus \tilde{A}]^{<\omega}$ so that*

$$|B \setminus \bigcup\{N(x, i) : x \in a, i \in I\}| < \kappa$$

then there is a λ-configuration $\mathcal{X}, \mathcal{Y}$ in colours I so that $\bigcup \mathcal{X} \subseteq A$.

Proof. We build the sequences $\mathcal{X} = \{a_\xi : \xi < \lambda\}$ and $\mathcal{Y} = \{y_\xi : \xi < \lambda\}$ inductively so that

$$|B \setminus \bigcup\{N(x, i) : x \in a_\xi, i \in I\}| < \kappa$$

for all $\xi < \lambda$. Given $\{a_\xi : \xi < \zeta\}$ and $\{y_\xi : \xi < \zeta\}$ we set $\tilde{A} = \bigcup\{a_\xi : \xi < \zeta\}$. Our assumption gives a finite set $a_\zeta \in [A \setminus \tilde{A}]^{<\omega}$ so that

$$|B \setminus \bigcup\{N(x, i) : x \in a_\zeta, i \in I\}| < \kappa.$$

As $X_\zeta = \bigcup\{a_\xi : \xi \leq \zeta\}$ has size $< \lambda \leq cf(\kappa)$, X_ζ is contained in an initial segment of the $H_{\kappa,\kappa}$ ordering. In turn,

$$|N[X_\zeta]| = \kappa.$$

Finally, as $|X_\zeta| < \kappa$, the set

$$Y_\zeta = \{y \in N[X_\zeta] : \forall \xi \leq \zeta : y \in \bigcup\{N(x, i) : x \in a_\xi, i \in I\}\}$$

has size κ. Picking $y_\zeta \in Y_\zeta \setminus \{y_\xi : \xi < \zeta\}$ finishes the proof. □

Claim 3.2.18. *Suppose that c is an r-edge colouring of a graph $G = (V, E)$ of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B). Let $I \subseteq r$ and $X \subseteq A$. If*

$$|B \setminus \bigcup\{N(x, i) : x \in a, i \in I\}| = \kappa$$

for all $a \in [X]^{<\omega}$ then there is a partition $\{X_j : j \in r \setminus I\}$ of X so that

$$|N(x, j) \cap N(x', j) \cap B| = \kappa$$

for all $x, x' \in X_j$ *and* $j \in r \setminus I$.

In particular, the sets X_j *are* κ*-unseparable in colour* j *in* $X_j \cup B$ *and if* $B \subseteq X$ *then there is* $j \in I \setminus r$ *so that* X_j *is* κ*-connected in colour* j.

Proof. Take a uniform ultrafilter U on B so that

$$B \setminus \bigcup \{N(x,i) : x \in a, i \in I\} \in U$$

for all $a \in [X]^{<\omega}$. Define $X_j = \{x \in X : N(x,j) \in U\}$ for $j < r$ and note that $X_j = \emptyset$ if $j \in I$ while $\{X_j : j \in r \setminus I\}$ partitions X.

It is clear that

$$|N(x,j) \cap N(x',j) \cap B| = \kappa$$

for all $x, x' \in X_j$ and $j \in r \setminus I$ and hence X_j is κ-unseparable in colour j. Furthermore, if $B \subseteq X$ then there is a $j \in r \setminus I$ so that $X_j \cap B \in U$ and hence X_j is κ-connected in j as

$$|N(x,j) \cap N(x',j) \cap X_j| = \kappa$$

for all $x, x' \in X_j$. $\square$

Claim 3.2.19. *Suppose that* H *is of type* $H_{\kappa,\kappa}$ *with classes* A, B *and* $\lambda < \kappa$. *If there is no* λ*-configuration* $\mathcal{X}, \mathcal{Y}$ *with* $\bigcup \mathcal{X} \subseteq A$ *then there is* $\tilde{A} \in [A]^{<\kappa}$ *so that*

$$|B \setminus \bigcup \{N(x,i) : x \in a, i \in I\}| = \kappa$$

for all $a \in [A \setminus \tilde{A}]^{<\omega}$.

Proof. First, suppose that $\kappa = cf(\kappa)$. Apply Claim 3.2.17 to the graph H and $\lambda = \kappa$ and find $\tilde{A} \in [A]^{<\kappa}$ so that

$$|B \setminus \bigcup \{N(x,i) : x \in a, i \in I\}| = \kappa$$

for all $a \in [A \setminus \tilde{A}]^{<\omega}$.

Second, suppose that $\kappa > cf(\kappa)$ and fix an increasing cofinal sequence of regular cardinal $(\kappa_\alpha)_{\alpha < cf(\kappa)}$ in κ so that $\kappa_0 > \lambda$. Let H_α denote $H \restriction \kappa_\alpha$; H_α is a graph of type $H_{\kappa_\alpha,\kappa_\alpha}$ and let A_α, B_α denote the two classes. Note that H_α still has no λ-configuration in colours I and hence we can apply Claim 3.2.17 to the graph H_α with $\lambda < \kappa_\alpha$: there is $\tilde{A}_\alpha \in [A_\alpha]^{<\lambda}$ so that

$$|B_\alpha \setminus \bigcup \{N(x,i) : x \in a, i \in I\}| = \kappa_\alpha$$

for all $a \in [A_\alpha \setminus \tilde{A}_\alpha]^{<\omega}$.

Let $\tilde{A} = \bigcup \{\tilde{A}_\alpha : \alpha < cf(\kappa)\}$ and note that $|\tilde{A}| \leq cf(\kappa) \cdot \lambda < \kappa$. Now, if $a \in [A \setminus \tilde{A}]^{<\omega}$ then $a \subseteq A_\alpha \setminus \tilde{A}_\alpha$ for any large enough $\alpha < cf(\kappa)$ and hence

$$|B_\alpha \setminus \bigcup \{N(x,i) : x \in a, i \in I\}| = \kappa_\alpha$$

for any large enough $\alpha < cf(\kappa)$. In turn

$$|B \setminus \bigcup \{N(x,i) : x \in a, i \in I\}| = \kappa.$$

□

Proof of Lemma 3.2.13. Suppose that condition (a) fails. In particular, for all $i \in I$ there is $\lambda_i < \kappa$ and $X_i^* \subset A$ of size less than κ so that there is no path of colour i concentrated on X and order type λ_i disjoint from X_i^*. Let $\lambda = \max\{\lambda_i : i \in I\}$ and $X^* = \bigcup\{X_i^* : i \in I\}$. Now, there is no path of colour $i \in I$ and of order type λ in $X \setminus X^*$ concentrated on X.

Now find a graph H of type $H_{\kappa,\kappa}$ in G with main class $X \setminus X^*$ and second class B'; this can be done by Observation 3.2.4. As $(IH)_{\lambda,|I|}$ holds, Claim 3.2.15 implies that there is no λ-configuration $\mathcal{X}, \mathcal{Y}$ with $\bigcup \mathcal{X} \subseteq X \setminus X^*$.

Apply Claim 3.2.19 in H and find $\tilde{A} \in [X \setminus X^*]^{<\kappa}$ so that

$$|B' \setminus \bigcup\{N(x,i) : x \in a, i \in I\}| = \kappa$$

for all $a \in [X \setminus (X^* \cup \tilde{A})]^{<\omega}$. Hence Claim 3.2.18 applied to $X \setminus (X^* \cup \tilde{A})$ provides the desired partition and hence clause (b) of Lemma 3.2.13.

□

3.2.4 The existence of monochromatic paths

We arrived at our first main result which shows, together with Lemma 3.1.8, the existence of large monochromatic paths in edge coloured graphs of type $H_{\kappa,\kappa}$:

Theorem 3.2.20. *$(IH)_\kappa$ holds for all infinite κ. In particular, if G is a graph of type $H_{\kappa,\kappa}$ with a finite-edge colouring then we can find a monochromatic path of size κ concentrated on the main class of G.*

Proof. We prove $(IH)_{\kappa,r}$ by induction on κ and $r \in \omega$. $(IH)_\omega$ holds by Theorem 1.3.7 so we suppose that $\kappa > \omega$. Also, $(IH)_{\kappa,1}$ holds by Observation 3.2.7.

Now fix an r-edge colouring of a graph G of type $H_{\kappa,\kappa}$ with $H_{\kappa,\kappa}$-decomposition (A, B).

First, we can suppose that any $X \in [A]^\kappa$ satisfies condition (3) of Lemma 3.1.8 *in all colours*. Indeed, given X we can find a graph H_0 of type $H_{\kappa,\kappa}$ in G with main class X (by applying Observation 3.2.4). Given any colour $i < r$, Lemma 3.2.8 applied to H_0 and colour i tells us that if X fails condition (3) of Lemma 3.1.8 in colour i then we can find a graph H_1 of type $H_{\kappa,\kappa}$ (with main class X minus a set of size $< \kappa$) which is only coloured by $r \setminus \{i\}$. Hence we can apply the inductive hypothesis $(IH)_{\kappa,r-1}$ to H_1 which finishes the proof.

Now, find a *maximal* $I \subseteq r$ so that there is $X \in [A]^\kappa$ such that X is κ-unseparable in all colours $i \in I$. Fix such an I and X. The following claim finishes the proof.

Claim 3.2.21. *There is $i \in I$ such that $\spadesuit_{\kappa,i}$ holds for X.*

Proof. Suppose that X fails $\spadesuit_{\kappa,i}$ for all $i \in I$. If $|I| < r$ then apply Lemma 3.2.13 in G to the set X and set of colours I. As X fails $\spadesuit_{\kappa,i}$ for all $i \in I$, condition (b) of Lemma 3.2.13 must hold; in turn, there is a colour $j \in r \setminus I$ and a set $X_j \in [X]^\kappa$ so that X_j is κ-unseparable in colour j as well. The fact that X_j is κ-unseparable in each colour $i \in I \cup \{j\}$ contradicts the maximality of I.

Hence $I = r$ must hold. Now, for each $i < r$ there is $\lambda_i < \kappa$ and $A_i^* \subset A$ of size less than κ so that there is no path of colour i concentrated on X which has order type λ_i and is disjoint from A_i^*. Let $\lambda^* = \max\{\lambda_i : i < r\}$ and $A^* = \bigcup\{A_i^* : i < r\}$. Now, there is no path of colour $i < r$ and of order type

λ^* which is concentrated on X and is disjoint from A^*. There is a graph H of type $H_{\kappa,\kappa}$ in G with main class $X \setminus A^*$ (by Observation 3.2.4) and the initial segment $H \restriction \lambda^*$ is of type H_{λ^*,λ^*}. As $(IH)_{\lambda^*,r}$ holds, we can find a path of type λ^* in $H \restriction \lambda^*$ which is concentrated on the main class and hence on X. This path is also disjoint from A^* which contradicts our previous assumption. □

□

3.3 The first decomposition theorem

Our goal now is to prove a path decomposition result for a large class of bipartite graphs which contains $H_{\kappa,\kappa}$.

Definition 3.3.1. *Suppose that $G = (V, E)$ is a graph, $A \subseteq V$ and κ is a cardinal. We say that A is $(\mathcal{A}, \kappa)$-centered (in G) iff $\mathcal{A} = \{(A^i_\alpha)_{\alpha<\lambda_i} : i \in I\}$ is a finite set of increasing covers of A and*

$$|N_G[\bigcap_{i\in I} A^i_{\alpha_i}]| \geq \kappa$$

for all $(\alpha_i)_{i\in I} \in \Pi_{i\in I}\lambda_i$.

In this section, $\mathcal{A}$ will always denote a finite set of increasing families (indexed by I) and $\overrightarrow{\lambda} = (\lambda_i)_{i\in I}$ denotes the length of these families.

Given $\mathcal{A}$ and $\overrightarrow{\alpha} = (\alpha_i)_{i\in I} \in \Pi\overrightarrow{\lambda}$ we will write $[\overrightarrow{\alpha}]_\mathcal{A}$ for $\bigcap_{i\in I} A^i_{\alpha_i}$. We call sets of the form $[\overrightarrow{\alpha}]_\mathcal{A}$ an $\mathcal{A}$-box. Furthermore, $\overrightarrow{\alpha} \leq \overrightarrow{\beta}$ will stand for $\alpha_i \leq \beta_i$ for all $i \in I$.

Note that if A is $(\emptyset, \kappa)$-centered then $|N_G[A]| = \kappa$. Also, the main class of a graph G of type $H_{\kappa,\kappa}$ is clearly $(\mathcal{A}, \kappa)$-centered where $\mathcal{A}$ is a single increasing cover formed by the initial segments of the $H_{\kappa,\kappa}$ ordering.

Our final goal in this section is to prove the following:

Theorem 3.3.2. *Suppose that $G = (V, E)$ is a bipartite graph on classes A, B where $|A| = \kappa$. Suppose that A is $(\mathcal{A}, \kappa)$-centered for some $\mathcal{A}$. Then for any finite edge colouring of G, A is covered by disjoint monochromatic paths of different colours.*

We start with basic observations:

Observation 3.3.3. *Suppose that A is $(\mathcal{A}, \kappa)$-centered in a graph G and $\overrightarrow{\alpha}, \overrightarrow{\beta} \in \Pi\overrightarrow{\lambda}$.*

1. *If $\overrightarrow{\alpha} \leq \overrightarrow{\beta}$ then $[\overrightarrow{\alpha}]_\mathcal{A} \subseteq [\overrightarrow{\beta}]_\mathcal{A}$ and hence $N_G[[\overrightarrow{\beta}]_\mathcal{A}] \subseteq N_G[[\overrightarrow{\alpha}]_\mathcal{A}]$;*
2. *$N_G[[\overrightarrow{\gamma}]_\mathcal{A}] \subseteq N_G[[\overrightarrow{\alpha}]_\mathcal{A}] \cap N_G[[\overrightarrow{\beta}]_\mathcal{A}]$ for $\gamma = \max_\leq\{\overrightarrow{\alpha}, \overrightarrow{\beta}\}$;*
3. *for every finite $F \subseteq A$ there is an $\mathcal{A}$-box Z covering F.*

In particular, any two points of A are joined by κ-many disjoint paths of length 2 and hence A is κ-unseparable.

Given a set of increasing covers $\mathcal{A} = \{(A^i_\alpha)_{\alpha<\lambda_i} : i \in I\}$ of A and $X \subseteq A$ we write $\mathcal{A} \restriction X$ for $\{(A^i_\alpha \cap X)_{\alpha<\lambda_i} : i \in I\}$.

Observation 3.3.4. *Suppose that A is $(\mathcal{A}, \kappa)$-centered in a graph G. Let $X \subseteq A$, $\overrightarrow{\alpha} \in \overrightarrow{\lambda}$ and H denote the subgraph in G spanned by $X \cup N_G[[\overrightarrow{\alpha}]_\mathcal{A}]$. Then X is $(\mathcal{A} \restriction X, \kappa)$-centered in H.*

Observation 3.3.5. *Suppose that* $(\tilde{A}^i_\alpha)_{\alpha<\tilde{\lambda}_i}$ *is a cofinal subsequence of* $(A^i_\alpha)_{\alpha<\lambda_i}$ *for each* $i \in I$*. Let* $\mathcal{A}$ *and* $\tilde{\mathcal{A}}$ *denote* $\{(A^i_\alpha)_{\alpha<\lambda_i} : i \in I\}$ *and* $\{(\tilde{A}^i_\alpha)_{\alpha<\tilde{\lambda}_i} : i \in I\}$ *respectively. Then a set of vertices* A *in a graph* $G = (V, E)$ *is* $(\mathcal{A}, \kappa)$*-centered iff* $(\tilde{\mathcal{A}}, \kappa)$*-centered.*

In particular, we can always suppose that $\lambda_i = cf(\lambda_i)$*, each cover is strictly increasing and hence* $\lambda_i \leq |A|$*.*

We say that a set of vertices $Y \subseteq V$ is $(\mathcal{A}, \kappa)$-dense iff

$$|Y \cap N_G[[\overrightarrow{\alpha}]_{\mathcal{A}}]| \geq \kappa$$

for all $\overrightarrow{\alpha} \in \Pi\overrightarrow{\lambda}$.

Observation 3.3.6. *Suppose that* A *is* $(\mathcal{A}, \kappa)$*-centered in a graph* G *and* $Y \subseteq V$ *is* $(\mathcal{A}, \kappa)$*-dense. Then*

1. $Y \cap N_G[[\overrightarrow{\alpha}]_{\mathcal{A}}]$ *is* $(\mathcal{A}, \kappa)$*-dense for all* $\overrightarrow{\alpha} \in \Pi\overrightarrow{\lambda}$*,*
2. *for any* $X \subseteq A$*,* X *is* $(\mathcal{A} \restriction X, \kappa)$*-centered in* $G \restriction (X \cup Y)$*.*

Our first non-trivial result connects the previously developed theory of $H_{\kappa,\kappa}$ to this new notion of $(\mathcal{A}, \kappa)$-centered subsets.

Lemma 3.3.7. *Suppose that* $G = (V, E)$ *is a bipartite graph on classes* A, B *where* $|A| = \kappa$*, and* A *is* $(\mathcal{A}, \kappa)$*-centered for some* $\mathcal{A}$*. Then there is a copy* H *of the graph* $H_{\kappa,\kappa}$ *with main class* $X \subseteq A$*.*

Proof. We can suppose that $\lambda_i = cf(\lambda_i) \leq \kappa$ for all $i \in I$ by Observation 3.3.5. Find a maximal $J \subseteq I$ such that there is $\alpha_j < \lambda_j$ for $j \in J$ so that $X_{-1} = \bigcap_{j\in J} A^j_{\alpha_j}$ has size κ. Note that J might be empty in which case $X_{-1} = A$. Note that $X_{-1} = \bigcup\{X_{-1} \cap A^i_\alpha : \alpha < \lambda_i\}$ is a union of sets of size $< \kappa$ and hence $cf(\kappa) \leq \lambda_i = cf(\lambda_i)$ for all $i \in I \setminus J$. Without loss of generality, $I \neq J$ otherwise $K_{\kappa,\kappa}$ embeds into G. Let us fix J, α_j and $A^j_{\alpha_j}$ as above.

First, suppose that κ is a limit cardinal and take a strictly increasing cofinal sequence $(\kappa_\xi)_{\xi<cf(\kappa)}$ in κ. Now inductively find $(\alpha_i(\xi))_{i\in I\setminus J} \in \Pi_{i\in I\setminus J}\lambda_i$ for $\xi < cf(\kappa)$ so that $(\alpha_i(\xi))_{i\in I\setminus J} \leq (\alpha_i(\zeta))_{i\in I\setminus J}$ and

$$X_\xi = X_{-1} \cap \bigcap_{i\in I\setminus J} A^i_{\alpha_i(\xi)} \text{ has size at least } \kappa_\xi$$

for all $\xi \leq \zeta < cf(\kappa)$.

Suppose $(\alpha_i(\xi))_{i\in I\setminus J}$ is constructed for $\xi < \zeta$. List $I \setminus J$ as $\{i_0, ..., i_m\}$. First, find $\alpha_{i_0}(\zeta) \in \lambda_{i_0} \setminus \sup\{\alpha_{i_0}(\xi) : \xi < \zeta\}$ such that

$$|X_{-1} \cap A^{i_0}_{\alpha_{i_0}(\zeta)}| \geq \kappa_\zeta^{+m}.$$

If we have $\alpha_{i_0}(\zeta), ..., \alpha_{i_{k-1}}(\zeta)$ for some $k < m$ so that

$$|X_{-1} \cap \bigcap_{l<k} A^{i_l}_{\alpha_{i_l}(\zeta)}| \geq \kappa_\zeta^{+m-k}$$

then find $\alpha_{i_k}(\zeta) \in \lambda_{i_k} \setminus \sup\{\alpha_{i_k}(\xi) : \xi < \zeta\}$ so that

$$|X_{-1} \cap \bigcap_{l\leq k} A^{i_l}_{\alpha_{i_l}(\zeta)}| \geq \kappa_\zeta^{+m-k-1}.$$

Let $X = \bigcup\{X_\xi : \xi < cf(\kappa)\}$ and note that X_ξ has size $< \kappa$ and $|N[X_\xi]| = \kappa$ since X_ξ is an $\mathcal{A}$-box for each $\xi < cf(\kappa)$. Observation 3.2.4 can be applied now to find a copy H of $H_{\kappa,\kappa}$ with main class X.

If $\kappa = \mu^+$ we inductively find $(\alpha_i(\xi))_{i\in I\setminus J} \in \Pi_{i\in I\setminus J}\lambda_i$ for $\xi < cf(\kappa)$ so that

$$X_\xi = X_{-1} \cap \bigcap_{i\in I\setminus J} A^i_{\alpha_i(\xi)} \text{ has size } \mu$$

and $X_\xi \subsetneq X_\zeta$ for all $\xi \le \zeta < \kappa$. First, note that $\lambda_i = \kappa$ for all $i \in I \setminus J$. As before, suppose $(\alpha_i(\xi))_{i\in I\setminus J}$ is constructed for $\xi < \zeta$ and list $I \setminus J$ as $\{i_0, ..., i_m\}$. Fix $x \in X_{-1} \setminus \bigcup\{X_\xi : \xi < \zeta\}$. Suppose we have $\alpha_{i_0}(\zeta), ..., \alpha_{i_{k-1}}(\zeta)$ for some $k < m$ so that

$$|X_{-1} \cap \bigcap_{l<k} A^{i_l}_{\alpha_{i_l}(\zeta)}| = \mu$$

and $x \in X_{-1} \cap \bigcap_{l<k} A^{i_l}_{\alpha_{i_l}(\zeta)}$. We claim that there is $\alpha_{i_k}(\zeta) \in \kappa \setminus \sup\{\alpha_{i_k}(\xi) : \xi < \zeta\}$ so that

$$|X_{-1} \cap \bigcap_{l\le k} A^{i_l}_{\alpha_{i_l}(\zeta)}| = \mu$$

and $x \in X_{-1} \cap \bigcap_{l\le k} A^{i_l}_{\alpha_{i_l}(\zeta)}$. Indeed, we cannot write a set of size μ as an increasing union of μ^+ sets of size $< \mu$.

Finally, let $X = \bigcup\{X_\xi : \xi < \kappa\}$. As before, X_ξ has size $< \kappa$ and $|N[X_\xi]| = \kappa$ for each $\xi < \kappa$. Hence Observation 3.2.4 can be applied to find a copy H of $H_{\kappa,\kappa}$ with main class X. □

The next lemma shows that the property of being "$(\mathcal{A}, \kappa)$-centered for some $\mathcal{A}$" is inherited by subgraphs in a strong sense.

Lemma 3.3.8. *Suppose that $G = (V, E)$ is a bipartite graph on classes A, B and A is $(\mathcal{A}, \kappa)$-centered for some $\mathcal{A}$. Suppose that H is a subgraph of G such that*

$$|N_G[x] \setminus N_H[x]| < \kappa$$

for all $x \in X = V(H) \cap A$. Then there is a finite $\mathcal{A}' \supseteq \mathcal{A} \restriction X$ so that X is $(\mathcal{A}', \kappa)$-centered in H.

Proof. We define $\mathcal{A}'$ by extending $\mathcal{A} \restriction X$ with at most two new covers depending on the size of X and on κ being a limit or successor cardinal.

First, if X happens to have size κ then let $(X^0_\alpha)_{\alpha<cf(\kappa)}$ be an increasing sequence of subsets of X of size less than κ with union X. We put $(X^0_\alpha)_{\alpha<cf(\kappa)}$ into $\mathcal{A}'$ if $|X| = \kappa$.

Second, if κ is a limit cardinal then let us take a strictly increasing cofinal sequence $(\kappa_\alpha)_{\alpha<cf(\kappa)}$ in κ and let

$$X^1_\alpha = \{x \in X : |N_G[x] \setminus N_H[x]| \le \kappa_\alpha\}$$

for $\alpha < cf(\kappa)$. We put $(X^1_\alpha)_{\alpha<cf(\kappa)}$ into $\mathcal{A}'$ as well if κ is a limit.

Let us show that $\mathcal{A}'$ works. If $Z \subseteq X$ is an $\mathcal{A}'$-box then $|Z| < \kappa$ and there is $\lambda < \kappa$ such that $|N_G[x] \setminus N_H[x]| \le \lambda$ for all $x \in Z$. In particular

$$|\bigcup_{x\in Z} N_G[x] \setminus N_H[x]| \le |Z| \cdot \lambda < \kappa.$$

Also, $|N_G[Z]| = \kappa$ as Z is contained in an $\mathcal{A}$-box. Hence the set

$$N_H[Z] = N_G[Z] \setminus \big(\bigcup_{x \in Z} N_G[x] \setminus N_H[x] \big)$$

has size κ.

$\square$

Lemma 3.3.8 is the reason we work with this new class of bipartite graphs instead of $H_{\kappa,\kappa}$. Note that if X is a subset of the main class of $H_{\kappa,\kappa}$ then X is not necessarily covered by a subgraph isomorphic to $H_{\lambda,\lambda}$ for some $\lambda \leq \kappa$.

The next lemma is our final preparation to the proof of Theorem 3.3.2.

Lemma 3.3.9. *Suppose that $G = (V, E)$ is a bipartite graph on classes A, B where $|A| = \kappa$ and A is $(\mathcal{A}, \kappa)$-centered for some $\mathcal{A}$. Let c be a finite edge colouring of G and suppose that G_0 is a subgraph of G with classes $V(G_0) \cap A = A_0$ and $V(G_0) \cap B = B_0$. If*

1. *$|A_0| = \kappa$ and B_0 is $(\mathcal{A}, \kappa)$-dense in G, and*
2. *$|\operatorname{ran}(c \restriction E(G_0))|$ is minimal among subgraphs of G satisfying (1)*

then

3. *for every $i \in \operatorname{ran}(c \restriction E(G_0))$ and every $X \in [A_0]^\kappa$ there is a set of κ independent edges $\{\{x_\alpha, y_\alpha\} : \alpha < \kappa\} \subseteq c^{-1}(i)$ so that $\{x_\alpha : \alpha < \kappa\} \subseteq X$ and $\{y_\alpha : \alpha < \kappa\}$ is $(\mathcal{A}, \kappa)$-dense in G.*

Proof. Suppose $\mathcal{A} = \{(A^i_\alpha)_{\alpha<\lambda_i} : i \in I\}$ and $\overrightarrow{\lambda} = (\lambda_i)_{i \in I}$ as before. Again, we can suppose that $\Pi \overrightarrow{\lambda}$ has size $\leq \kappa$ by Observation 3.3.5. Take a subgraph G_0 of G which satisfies (1) and suppose that (3) fails; we will show that $|\operatorname{ran}(c \restriction E(G_0))|$ is not minimal i.e. (2) fails.

Let $i \in \operatorname{ran}(c \restriction E(G_0))$ and $X \in [A_0]^\kappa$ witness that condition (3) fails. Enumerate $\Pi \overrightarrow{\lambda}$ as $\{\overrightarrow{\alpha}(\xi) : \xi < \kappa\}$ such that each $\overrightarrow{\alpha} \in \Pi \overrightarrow{\lambda}$ appears κ times. Start inductively building independent edges $\{\{x_\xi, y_\xi\} : \xi < \zeta\} \subseteq c^{-1}(i)$ from X so that $y_\xi \in B_0 \cap N_G[[\overrightarrow{\alpha}(\xi)]_\mathcal{A}]$. There must be a $\zeta < \kappa$ such that we cannot pick $\{x_\zeta, y_\zeta\}$. That is, every edge from $X \setminus \{x_\xi : \xi < \zeta\}$ to $B_0 \cap N_G[[\overrightarrow{\alpha}(\zeta)]_\mathcal{A}] \setminus \{y_\xi : \xi < \zeta\}$ is not coloured i. Let $A_1 = X \setminus \{x_\xi : \xi < \zeta\}$ and $B_1 = B_0 \cap N_G[[\overrightarrow{\alpha}(\zeta)]_\mathcal{A}] \setminus \{y_\xi : \xi < \zeta\}$. It is easy to see that $G_1 = G_0 \restriction A_1 \cup B_1$ satisfies (1); indeed, A_1 has size κ and Observation 3.3.6 implies that B_1 is $(\mathcal{A}, \kappa)$-dense in G. Finally, $i \notin \operatorname{ran}(c \restriction E(G_1))$ implies $|\operatorname{ran}(c \restriction E(G_1))| < |\operatorname{ran}(c \restriction E(G_0))|$ and we are done.

$\square$

Proof of Theorem 3.3.2. We prove the statement by induction on $r \geq 1$ for every $G = (V, E)$, $\mathcal{A}$ and c simultaneously.

First, suppose $r = 1$. Lemma 3.3.7 implies that we can find a copy H of $H_{\kappa,\kappa}$ in G with main class $X \subseteq A$. Hence, by Theorem 3.2.20, there is a path P of size κ which is concentrated on X. As A is κ-unseparable (by Observation 3.3.3) we can cover A by a single path in G using Lemma 3.1.3.

Now, suppose we proved the statement for $r - 1$ and fix $G = (V, E)$, $\mathcal{A}$ and an r-edge colouring c. We will show that there is a colour $i < r$ and a path P of colour i in G such that $A \setminus P$ is one class of a bipartite subgraph G_1 of G so that

(i) $V(G_1) \cap P = \emptyset$,

(ii) $i \notin \operatorname{ran}(c \restriction E(G_1))$,

(iii) $A \setminus P$ is $(\mathcal{A}', \kappa)$-centered in G_1 for some finite $\mathcal{A}' \supseteq \mathcal{A}$.

Once we find such a path P and subgraph G_1, applying the inductive hypothesis finishes the proof.

First, take a subgraph G_0 of G with classes $V(G_0) \cap A = A_0$ and $V(G_0) \cap B = B_0$ such that

1. $|A_0| = \kappa$ and B_0 is $(\mathcal{A}, \kappa)$-dense in G, and

2. $|\operatorname{ran}(c \restriction E(G_0))|$ is minimal among subgraphs of G satisfying (1).

Find a partition of B_0 into B_0^0 and B_0^1 so that both sets are $(\mathcal{A}, \kappa)$-dense in G. Let $G_0^l = G_0 \restriction (A_0 \cup B_0^l)$ and note that $\operatorname{ran}(c \restriction E(G_0^1)) = \operatorname{ran}(c \restriction E(G_0))$ by (2). Hence, by Lemma 3.3.9, for every $i \in \operatorname{ran}(c \restriction E(G_0))$ and every $X \in [A_0]^\kappa$ there is a set of κ independent edges $\{\{x_\alpha, y_\alpha\} : \alpha < \kappa\} \subseteq c^{-1}(i)$ so that $\{x_\alpha : \alpha < \kappa\} \subseteq X$ and $\{y_\alpha : \alpha < \kappa\} \subseteq B_0^1$ is $(\mathcal{A}, \kappa)$-dense in G.

Now, embed a copy H of $H_{\kappa,\kappa}$ in G_0^0 using Lemma 3.3.7. By Theorem 3.2.20, we can find $i < r$ and a set X in the main class of H which satisfies all three conditions of Lemma 3.1.8 in colour i. By (2), there is a set of κ independent edges $\{\{x_\alpha, y_\alpha\} : \alpha < \kappa\} \subseteq c^{-1}(i)$ in G_0^1 so that $\{x_\alpha : \alpha < \kappa\} \subseteq X$ and $Y = \{y_\alpha : \alpha < \kappa\} \subseteq B_0^1$ is $(\mathcal{A}, \kappa)$-dense in G.

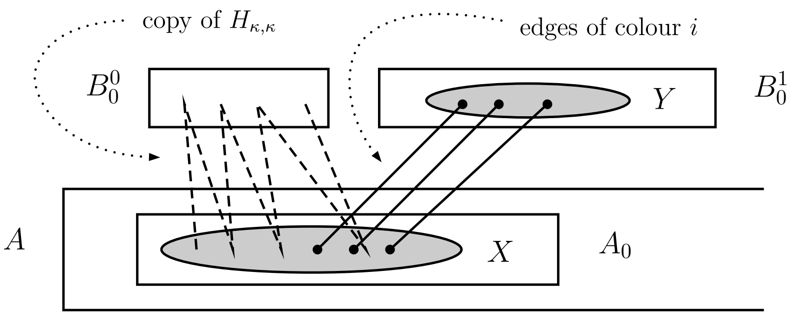


Figure 3.4: Preparing the cover of A.

Let

$$\bar{X} = X \cup \{x \in A : |N_G(x, i) \cap Y| = \kappa\}.$$

Note that $\bar{X}$ is still κ-unseparable in G.

Claim 3.3.10. *There is $Y_1 \in [Y]^\kappa$ so that Y_1 is $(\mathcal{A}, \kappa)$-dense in G and*

$$|N_G(x, i) \cap Y \setminus Y_1| = \kappa$$

for all $x \in \bar{X} \setminus X$.

Proof. The proof goes by an easy induction of length κ. $\square$

Note that $\bar{X}$ still satisfies all three condition of Lemma 3.1.8 in $V \setminus Y_1$ and $|N_G(x, i) \cap Y_1| < \kappa$ for all $x \in A \setminus \bar{X}$. Now find a path P of colour i in $V \setminus Y_1$ which covers $\bar{X}$; this can be done by Lemma 3.1.8. Note that $A \setminus P$ is $(\mathcal{A} \restriction A \setminus P, \kappa)$-centered in $G \restriction (A \setminus P \cup Y_1)$ and the subgraph

$$G_1 = (A \setminus P \cup Y_1, c^{-1}(r \setminus \{i\}))$$

satisfies the assumptions of Lemma 3.3.8. In particular, $A \setminus P$ is $(\mathcal{A}', \kappa)$-centered for some finite $\mathcal{A}' \supseteq \mathcal{A}$ in G_1. This finishes the proof.

□

3.4 The main decomposition theorem

At this point, it would be rather easy to show (using Theorem 3.3.2) that every κ-complete graph is covered by $2r$ (not necessarily disjoint) monochromatic paths. However, we prove the following much stronger theorem which is the main result of this chapter:

Theorem 3.4.1. *Suppose that c is a finite-edge colouring of a κ-complete graph $G = (V, E)$. Then the vertices can be partitioned into disjoint monochromatic paths of different colours.*

Proof. We can suppose $\kappa > \omega$. First, note that any κ-complete graph $G = (V, E)$ is actually $|V|$-complete; thus it suffices to prove the theorem for κ-complete graphs of size κ. The next arguments will be reminiscent of the proof of Theorem 3.2.20.

Claim 3.4.2. *Suppose that c is an r-edge colouring of G with $r \in \omega$. Then there is $A \in [V]^\kappa$ and $i < r$ so that A is κ-connected in colour $i < r$ and satisfies $\spadesuit_{\kappa,i}$ in $G \restriction A$ at the same time.*

Proof. Suppose there is no such A. By a finite induction, we construct sets $A_0 \supseteq A_1 \supseteq \dots$ of size κ and a 1-1 sequence $i_0, i_1, \dots$ in r so that A_k is κ-connected in colour i_k.

Suppose $k = 0$. As G is of type $H_{\kappa,\kappa}$ with main class V (see Observation 3.2.5) we can apply Claim 3.2.18 with $I = \emptyset$ and $X = V$. We find a colour i_0 and a set A_0 of size κ which is connected in colour i_0.

Suppose $k < r - 1$ and we defined A_k. As A_j must fail $\spadesuit_{\kappa,i_j}$ in $G \restriction A_j$ for all $j \leq k$, we have $A_j^* \in [A_j]^{<\kappa}$ and $\lambda_j < \kappa$ such that there is no path P in colour i_j in $G \restriction A_j \setminus A_j^*$ which is concentrated on A_j and has order type λ_j. Set $A^* = \bigcup\{A_j^* : j \leq k\}$ and $\lambda = \max\{\lambda_j : j \leq k\}$. Note that $H = G \restriction (A_k \setminus A^*)$ is of type $H_{\kappa,\kappa}$ with main class $A_k \setminus A^*$ and there is no λ-configuration in colours $I = \{i_j : j \leq k\}$ inside H. Indeed, otherwise Claim 3.2.15 would imply that there is a path of type λ in colour i_j inside in $G \restriction A_j \setminus A_j^*$ for some $j \leq k$ (recall that $A_k \setminus A^*$ is κ-unseparable in colour i_j in $G \restriction A_j \setminus A_j^*$). Hence, Claim 3.2.19 and 3.2.18 implies that we can find a set $A_{k+1} \in [A_k]^\kappa$ and colour $i_{k+1} \in r \setminus \{i_j : j \leq k\}$ so that A_{k+1} is κ-connected in colour i_{k+1}.

Suppose we defined A_{r-1}. By assumption, A_{r-1} fails $\spadesuit_{\kappa,i}$ in $G \restriction A_{r-1}$ for all $i < r$. However, Theorem 3.2.20 implies the existence of a monochromatic path of size κ in some colour $i < r$ which in turn implies that $\spadesuit_{\kappa,i}$ must hold for some $i < r$ by Observation 3.1.7.

□

Claim 3.4.3. *There are sets $A, Y \in [V]^\kappa$ and $i < r$ so that $Y \subseteq A$ and $A \setminus Z$ satisfies all three conditions of Lemma 3.1.8 in colour i in $G \restriction A \setminus Z$ for all $Z \subseteq Y$. Moreover, we can suppose that A is a* maximal *κ-connected subset.*

In particular, $A \setminus Z$ is a single path of colour i for *every choice of $Z \subseteq Y$* by Lemma 3.1.8.

Proof. This claim is proved by induction on r. If $r = 1$ then let $A = V$ and let $Y \subseteq A$ such that $A \setminus Y$ and Y has size κ. Given $Z \subset Y$, we know that $G \restriction (A \setminus Z)$ is κ-complete and hence of type $H_{\kappa,\kappa}$ with main class $A \setminus Z$. Hence, by Observation 3.2.7, $A \setminus Z$ satisfies all three conditions of Lemma 3.1.8 in $G \restriction (A \setminus Z)$.

Suppose that $r > 1$. Now, we can suppose that any set $X \in [V]^\kappa$ satisfies condition (3) of Lemma 3.1.8 in $G \restriction X$ in *all colours* $i < r$. Indeed, note that $G \restriction X$ is of type $H_{\kappa,\kappa}$ with main class X and suppose X fails condition (3) of Lemma 3.1.8 in $G \restriction X$ in some colour $i < r$. Now Lemma 3.2.8 implies that there is $\tilde{X} \in [X]^{<\kappa}$ so that $X \setminus \tilde{X}$ is covered by a subgraph H of $G \restriction X$ of type $H_{\kappa,\kappa}$ with main class $X \setminus \tilde{X}$ so that $i \notin \operatorname{ran}(c \restriction E(H))$. Without loss of generality $V(H) \cap \tilde{X} = \emptyset$ i.e. $V(H)$ is the main class of H. Hence Observation 3.2.5 implies that we can find a κ-complete subgraph G' in H; the inductive hypothesis can be applied to G' as $i \notin \operatorname{ran}(c \restriction E(G'))$.

Now, take $A \in [V]^\kappa$ which is a maximal κ-connected subset in some colour $i < r$ and satisfies $\spadesuit_{\kappa,i}$ in $G \restriction A$; this can be done by Claim 3.4.2. It is easy to see that we can find $Y \in [A]^\kappa$ so that $A \setminus Z$ is still κ-connected in colour i and satisfies $\spadesuit_{\kappa,i}$ in $G \restriction A \setminus Z$ for any $Z \subseteq Y$. Indeed, we construct Y by an induction of length $cf(\kappa)$: let $\{\kappa_\alpha : \alpha < cf(\kappa)\}$ be a cofinal sequence of cardinals in κ ($\kappa_\alpha = \lambda$ if $\kappa = \lambda^+$) and let $A_\alpha \in [A]^{\kappa_\alpha}$ increasing so that $A = \bigcup\{A_\alpha : \alpha < cf(\kappa)\}$. Define sets $Y_\alpha \in [A]^{\kappa_\alpha}$, $W_\alpha \in [A]^{\kappa_\alpha}$ for $\alpha < cf(\kappa)$ so that $Y_\alpha \cap W_\beta = \emptyset$ for all $\alpha, \beta < cf(\kappa)$ and

1. there are κ_α many disjoint paths of order type κ_α and colour i in $G \restriction W_\alpha$ concentrated on A, and
2. for any $u \neq v \in A_\alpha$, there are κ_α many disjoint paths of colour i from v to u in $W_\alpha \cup \{u, v\}$.

It is clear that $Y = \bigcup\{Y_\alpha : \alpha < cf(\kappa)\}$ is as desired. As $A \setminus Y$ satisfies (3) of Lemma 3.1.8, we are done.

$\square$

Find $A, Y \subset V$ and $i < r$ as in Claim 3.4.3 with A being a maximal κ-connected subset in colour i. Let $X = V \setminus \bar{A}$. Let H denote the bipartite subgraph of G on classes X, Y where $\{v, w\} \in E(H)$ iff $v \in Y, w \in X$ and $c(v, w) \neq i$. Note that

$$|Y \setminus N_H(x)| < \kappa \text{ for all } x \in X;$$

otherwise $\bar{A} \cup \{x\}$ is still κ-connected in colour i.

If K denotes the complete bipartite graph on classes X, Y then X is $(\emptyset, \kappa)$-centered in K. Furthermore, the subgraph H of K satisfies the conditions of Lemma 3.3.8 and hence there is a finite $\mathcal{A}'$ so that X is $(\mathcal{A}', \kappa)$-centered in H.

By Theorem 3.3.2, there is a set of disjoint monochromatic paths $\mathcal{Q}$ in H which covers X; recall that $i \notin \operatorname{ran}(c \restriction E(H))$ and hence none of the paths in $\mathcal{Q}$ has colour i. Note that $Z = \cup\mathcal{Q} \setminus X \subseteq Y$ and hence $V \setminus \cup\mathcal{Q} = \bar{A} \setminus Z$ satisfies all three conditions of Lemma 3.1.8 in colour i in $G \restriction (V \setminus \cup\mathcal{Q})$. In particular, $V \setminus \cup\mathcal{Q}$ is a single path in colour i and hence $\mathcal{Q} \cup \{P\}$ is a decomposition of $V(G)$ into disjoint monochromatic paths of different colours.

$\square$

3.5 Open problems

It is a natural question if one can extend our result to infinite complete bipartite graphs:

Conjecture 3.5.1. *Suppose that the edges of an infinite complete bipartite graph are coloured with $r \in \omega$ colours. Then we can partition the vertices into $2r - 1$ disjoint monochromatic paths.*

Note that Theorem 3.3.2 implies that we can find a cover (not necessarily disjoint) by $2r$ monochromatic paths. Also, the conjecture holds for the countably infinite case by Theorem 2.4.1.

3.5.1 Same problem, more colours

One can consider the monochromatic path decomposition problem when the edges of the complete graph are coloured with infinitely many colours. There is a simple limitation of proving a monochromatic path decomposition theorem, namely one might not be able to decompose the vertices into sets so that each set is connected in some colour. This problem was investigated by A. Hajnal, P. Komjáth, L. Soukup and I. Szalkai in [44]. Let's say that a *μ-decomposition* of an edge coloured graph is a partition of the vertices into μ sets so that each set is connected in some colour. The following was proved in [44]:

Theorem 3.5.2 ([44]).

1. *If $\mu < \omega$ and $\kappa \geq \omega$ then there is a μ-decomposition for every μ-edge colouring of K_κ.*
2. *Suppose GCH holds. If $cf(\kappa) = \mu^+$ then there is a μ-edge colouring of K_κ with no μ-decomposition.*
3. *Suppose MA_κ holds. Then there is an ω-decomposition for every ω-edge colouring of K_κ.*
4. *If $cf(\kappa) > 2^\mu$ then there is a μ-decomposition for every μ-edge colouring of K_κ.*
5. *It is consistent that 2^ω is arbitrarily large and there is an ω-edge colouring of K_{ω_1} with no ω-decomposition.*

A possible first step towards a general result could be looking at the following Ramsey-theoretic problem: let $\mathbf{P}$ denote the class of cardinals κ such that for every edge colouring $c : [\kappa]^2 \to \omega$ of K_κ there is a monochromatic path of size κ. It is easy to colour the edges of K_{ω_1} with ω colours without monochromatic cycles and hence $\omega_1 \notin \mathbf{P}$. Furthermore, note that if κ satisfies the partition relation $\kappa \to (\kappa)^2_\omega$ then $\kappa \in \mathbf{P}$ hence many large cardinals are in $\mathbf{P}$.

Problem 3.5.3. *Can we prove that $\mathbf{P}$ is non empty in ZFC? If so, what is $\min \mathbf{P}$?*

ω_2 or $\mathfrak{c}^+$ seem to be natural candidates for $\min \mathbf{P}$.

Chapter 4

The chromatic number and obligatory subgraphs

4.1 A brief history of the problem

A fundamental problem of graph theory asks how large chromatic number affects structural properties of a graph and in particular, is it true that a graph with large chromatic number has certain obligatory subgraphs? Let us review some of the most important results.

The first result in this area is due to W. Tutte, alias Blanche Descartes [10] (and independently A. Zykov [105]): a construction of triangle free graphs of arbitrary large finite chromatic number. This result was significantly extended by Paul Erdős:

Theorem 4.1.1 ([15]). *For any natural numbers $k, l \in \mathbb{N}$ there are graphs G of chromatic number $\geq k$ such that any cycle in G has size $\geq l$.*

Tutte's result extends to arbitrary *infinite* chromatic number [26] but can we generalize the Erdős' theorem to graphs with uncountable chromatic number? Note that there is no difficulty in constructing a graph G with $Chr(G) = \omega$ without small cycles; indeed, one takes disjoint copies of graphs G_k with chromatic number $\geq k$ without cycles of a given size.

In 1966 in their seminal paper, P. Erdős and A. Hajnal proved the following:

Theorem 4.1.2 ([20, Corollary 5.6]). *Every graph G with $Col(G) > \omega$ contains a copy of the complete bipartite graph K_{n,ω_1} for each $n \in \omega$.*

We present a simple proof of this theorem in Corollary 4.2.4. A particular consequence is that a copy of a four cycle embeds into G if $Chr(G) > \omega$ which is in striking contrast with the finite case and Erdős' result. We mention that it was already shown in [20] that one can find graphs G with uncountable chromatic number but without copies of $K_{\omega,\omega}$. What can be said about odd cycles then?

Theorem 4.1.3 ([20, Theorem 7.4]). *For each infinite κ and $j \in \omega$, there is a graph G of chromatic number and size κ such that G contains no cycles of length $2i + 1$ for $1 \leq i \leq j$.*

In particular, a finite graph H embeds into *every* graph G with uncountable chromatic/colouring number if and only if H is bipartite. However, any graph G with $Chr(G) > \omega$ contains all sufficiently large odd cycles [23, 97].

Yet another consequence of Theorem 4.1.2 is that every uncountably chromatic graph G contains an n-connected subgraph for each finite n (namely $K_{n,n}$). We will return to the connection of connectivity and chromatic number in Chapter 5 and 6.

There is a lot more that can be said about finite subgraphs of graphs with uncountable chromatic number, both results and open problems, but let us turn now to infinite obligatory subgraphs. A. Hajnal and P. Komjáth proved in 1983 the following

Theorem 4.1.4 ([42, Theorem 1]). *The graph $H_{\omega,\omega+1}$ embeds into every graph G with $Chr(G) > \omega$.*

Again, this result is included in our Corollary 4.2.4. Also, note that every graph G with $Chr(G) > \omega$ contains an infinite path (this was already proved in [20]) as $H_{\omega,\omega+1}$ contains an infinite path. We deal with generalizations of this fact in Section 4.3. Let us remark that Theorem 4.1.4 is sharp in the following sense: $H_{\omega,\omega+2}$ does not embed into every graph with uncountable chromatic number. This was first shown using CH in [42, Theorem 3], later in ZFC [64, Theorem 10] and we present another ZFC example in Section 6.5.

Let us recall a result of Komjáth from 1985 [56] which settles the *obligatory subgraph problem* for graphs with $Col(G) > \omega$.

Theorem 4.1.5 ([56, Theorem 3.1, Theorem 3.6]). *There are bipartite graphs $\Gamma_\omega, \Delta_\omega$ such that*

1. *Γ_ω is countable, embeds into every graph G with $Col(G) > \omega$ and Γ_ω contains each countable graph with this property;*
2. *Δ_ω has size $\aleph_1$, embeds into every graph G with $Col(G) > \omega$ and Δ_ω contains each graph sharing this property.*

The graphs $\Gamma_\omega, \Delta_\omega$ resemble $H_{\omega,\omega+1}$ in some sense but we omit their definitions as we will not use this result. We lack such satisfactory results for the chromatic number.

Also in [56], it is proved that there are two simple operations which result in obligatory subgraphs when applied to obligatory ones: if Γ embeds into every graph G with uncountable chromatic number then so does $\aleph_1$ disjoint copies of Γ and the graph $\Gamma \cup \bigcup\{V_x : x \in V(\Gamma)\}$ where V_x are disjoint sets of size $\aleph_1$ and $\{x, y\}$ is an edge for all $x \in V(\Gamma), y \in V_x$.

Lastly, we remark that work has already been done on the hard task of characterizing obligatory substructures of uncountably chromatic triple systems [59, 43, 66].

This short introduction barely touched the immense and fascinating theory of graphs with uncountable chromatic number (not to speak of results on chromatic number of finite graphs); we return to some open problems in Section 4.4. Finally, we refer the interested reader to the recent surveys by P. Komjáth [61, 63] and S. Todorcevic [102] on various topics concerning infinite graphs and chromatic number problems.

4.2 Classical results on obligatory subgraphs

Our goal in this section is to review the most important results on obligatory subgraphs of graphs with uncountable chromatic number. In doing so, we provide simplified proofs to several classical results from the literature.

Claim 4.2.1. *Suppose that $G = (V, E)$ is a graph, $N \prec H(\theta)$ with $G \in N$ and $x \in V \setminus N$. Then*

1. *$N_G[a]$ is uncountable for all $a \in [N_G(x) \cap N]^{<\omega}$,*

2. *if $N_G(x) \cap N$ is infinite then G contains a copy of K_{n,ω_1} for all $n \in \omega$, a copy of $H_{\omega,\omega+1}$ and an infinite ω_1-unseparable set.*

Proof. Note that $X = \bigcap\{N_G(y) : y \in a\} \in N$ for any finite $a \subseteq N_G(x) \cap N$. $|X| \geq \omega_1$ follows from $x \in X \setminus N$. This proves (1).

If $N_G(x) \cap N$ is infinite then we can choose $a \in [N_G(x) \cap N]^n$ for given $n \in \omega$ and (1) implies the existence of K_{n,ω_1} in G. To build a copy of $H_{\omega,\omega}$ in G we inductively pick vertices v_n, w_n so that $v_n \in N_G(x) \cap N \setminus \{v_m, w_m : m < n\}$ and $w_n \in N_G[\{v_m : m \leq n\}] \setminus \{w_m : m < n\}$. The set $\{v_n, w_n : n \in \omega\}$ is a copy of $H_{\omega,\omega}$.

Finally, the infinite set $N_G(x) \cap N$ is ω_1-unseparable as any two points are joined by uncountable many disjoint paths of length 2. □

Erdős and Kakutani [25, Theorem 1] proved that $\kappa = \omega_1$ is the only uncountable cardinal such that the edges of K_κ can be partitioned into countably many trees i.e. graphs without cycles. S. Todorcevic proves that $\kappa = \omega_1$ is the only uncountable cardinal such that the edges of K_κ can be partitioned into countably many graphs without infinite paths [101, Lemma 3.4.15 and 3.4.17]. Actually, the following holds:

Corollary 4.2.2. *$\kappa = \omega_1$ is the only uncountable cardinal such that the edges of K_κ can be partitioned into countably many graphs without a copy of $H_{\omega,\omega+1}$.*

Proof. If $\kappa = \omega_1$ then we refer to the above cited [25, Theorem 1]. Alternatively, take any colouring $f : [\omega_1]^2 \to \omega$ such that $f(\cdot, \beta) : \beta \to \omega$ is 1-1 for each $\beta < \omega_1$.

Now, fix a colouring $f : [\omega_2]^2 \to \omega$ and take an elementary submodel M of size ω_1 so that $f \in M$. Note that for any $\beta \in \omega_2 \setminus M$ there is $n \in \omega$ with

$$|\{\alpha \in M : f(\alpha, \beta) = n\}| \geq \omega.$$

Applying Claim 4.2.1 finishes the proof. □

Claim 4.2.3. *If $Col(G) > \omega$ then there is an elementary submodel M with $G \in M$ so that*

$$|N_G(x) \cap M| \geq \omega$$

for some $x \in V \setminus M$.

Proof. We can suppose that every $G' \subset G$ of size $< |G|$ has countable colouring number by restricting ourselves to a subgraph of G. Take a continuous chain of elementary submodels $(M_\alpha)_{\alpha<\kappa}$ covering G so that $G \in M_0$ and M_α has size $< |G|$. Suppose that $|N_G(x) \cap M_\alpha| < \omega$ for all $x \in V \setminus M_\alpha$ and we prove that G has colouring number $\leq \omega$. By assumption, there is a well ordering $\prec_\alpha$ on $V_\alpha = V \cap M_{\alpha+1} \setminus M_\alpha$ witnessing that $Col(G \restriction M_{\alpha+1} \setminus M_\alpha) \leq \omega$. Let $x \prec y$ iff $x \in V_\alpha, y \in V_\beta$ and $\alpha < \beta$ or $x, y \in V_\alpha$ and $x \prec_\alpha y$. It is clear that $\prec$ witnesses $Col(G) \leq \omega$. □

The following was first proved by Erdős, Hajnal [20, Corollary 5.6] and Hajnal, Komjáth [42, Theorem 1] respectively.

Corollary 4.2.4. *Every graph G with $Col(G) > \omega$ contains a copy of K_{n,ω_1} for all $n \in \omega$ and a copy of $H_{\omega,\omega+1}$.*

Proof. Apply Claims 4.2.3 and 4.2.1. $\square$

As we already mentioned, these results are sharp: there are graphs G with $Chr(G) > \omega$ so that G has no triangles nor copies of $H_{\omega,\omega+2}$; this was proved first in [42, Theorem 3] from CH, then [64, Theorem 10] in ZFC and we present another example in Section 6.5.

Also, from Claims 4.2.3 and 4.2.1 we get the following:

Corollary 4.2.5. *Every graph G with $Col(G) > \omega$ contains an infinite ω_1-unseparable subset.*

In Corollary 4.3.8, we show that there are graphs G with $Col(G) > \omega$ which contain no uncountable ω_1-unseparable subsets.

The following was proved by Erdős and Hajnal [20, Theorem 7.2]:

Theorem 4.2.6. *Every graph $G = (V, E)$ with $Chr(G) > \omega$ contains a graph H with $Chr(H) > \omega$ and minimal degree ω.*

Proof. We prove by contradiction: suppose $Chr(G) > \omega$ but the minimal degree of $G \restriction U$ is finite for every $U \subseteq V$ with $Chr(G \restriction U) > \omega$. Hence, we can build a well ordered set $W = \{v_\alpha : \alpha < \mu\}$ for some ordinal μ with the property that

$$|\{\beta \in \mu \setminus \alpha : v_\beta \in N_G(v_\alpha)\}| < \omega$$

for all $\alpha < \mu$ and $V \setminus W$ is countably chromatic. Indeed, we inductively pick vertices $v_\beta \in V \setminus \{v_\alpha : \alpha < \beta\}$ so that $|N_G(v_\beta) \cap V \setminus \{v_\alpha : \alpha < \beta\}| < \omega$ provided that the chromatic number of the induced subgraph on $V \setminus \{v_\alpha : \alpha < \beta\}$ is uncountable.

To reach a contradiction it suffices to show that $Chr(G \restriction W) \le \omega$. The next claim finishes the proof:

Claim 4.2.7. *Suppose that a graph G on a well ordered vertex set $W = \{v_\alpha : \alpha < \mu\}$ satisfies $|\{\beta \in \mu \setminus \alpha : v_\beta \in N_G(v_\alpha)\}| < \omega$ for all $\alpha < \mu$. Then there is another well order $\prec$ on W with the property that*

$$|\{w \in N_G(v) : w \prec v\}| < \omega$$

for all $v \in W$.

Indeed, $\prec$ witnesses that $Col(G \restriction W) \le \omega$ and hence $Chr(G \restriction W) \le \omega$ by Fact 1.4.3.

Proof. First, note that if $M \prec H(\Theta)$ is an elementary submodel and $G, \{v_\alpha : \alpha < \mu\} \in M$ then the finite set $\{v_\beta \in N_G(v_\alpha) : \beta \in \mu \setminus \alpha\}$ is an element and hence a subset of M for all $\alpha < \mu$ such that $v_\alpha \in M$.

Now take a sequence of countable elementary submodels $(M_\xi)_{\xi<\mu}$ covering W so that $(v_\alpha)_{\alpha<\mu}$, $G \in M_\xi$ for all $\xi < \mu$. Let $M_{<\zeta} = \bigcup\{M_\xi : \xi < \zeta\}$ for $\zeta < \mu$.

Fix $\alpha < \mu$. Note that

$$N_{G\restriction W}[v_\alpha] \cap M_{<\zeta} \subseteq \{v_\beta \in N_G(v_\alpha) : \beta \in \mu \setminus \alpha\}$$

whenever $v_\alpha \notin M_{<\zeta}$. Indeed, if $v_\beta \in N_G(v_\alpha) \cap M_\xi$ (for some $\xi < \zeta$) and $v_\alpha \notin M_\xi$ then $\beta > \alpha$ by the previous observation. Hence $N_{G\restriction W}[v_\alpha] \cap M_{<\zeta}$ is finite if $v_\alpha \notin M_{<\zeta}$.

Finally, define a well order $\prec$ on W so that

1. $W \cap M_\zeta \setminus M_{<\zeta}$ has order type $\leq \omega$, and
2. $v \in M_{<\zeta}$ and $w \in W \setminus M_{<\zeta}$ implies $v \prec w$

for all $\zeta < \mu$. It is clear that $\prec$ satisfies the required property. □

□

4.3 Paths and the chromatic number

Erdős and Hajnal proved that every graph G with $Col(G) > \omega$ contains an infinite path [20, Theorem 7.1]; note that this also follows from the fact that $H_{\omega,\omega}$ embeds into G. We wish to examine if this result extends to longer paths.

Proposition 4.3.1. *$H_{\omega,\omega}$ contains a path of order type ξ for all $\xi < \omega_1$.*

Proof. Suppose $H_{\omega,\omega}$ has classes A, B. We need the following

Observation 4.3.2. *There are pairwise disjoint $A_n \subset A$ and $B_n \subset B$ such that $A_n \cup B_n$ is isomorphic to $H_{\omega,\omega}$.*

We prove the proposition by induction on $\xi < \omega_1$ with the extra condition that if ξ is a successor ordinal then we can find a path of order type ξ which extends to a path terminating in a point of A; clearly, we can suppose that $\xi > \omega$. If we are done for ξ then take A_n, B_n as in the observation and find a path P of order type $\geq \xi$ in A_0, B_0, terminating in A_0 if ξ is a successor. If ξ is a limit then $B_0 \cap P$ is cofinal in P and so any point of A_1 is a valid continuation of P, i.e. $P^\frown(x)$ is a path of order type $\xi + 1$ for any $x \in A_1$. If ξ is successor then we can further extend to a path of order type $\xi + \omega$ using points for $A_1 \cup B_1$. Indeed, note that $B_i \subseteq^* N_G(x) \cap N_G(x')$[1] for all $x, x' \in A$ and $i < \omega$. So we choose $x_{\xi+2k} \in A_1$ for $k \in \omega$ and let $x_{\xi+2k+1} \in B_1 \cap N(x_{\xi+2k}) \cap N(x_{\xi+2k+2})$.

Now suppose ξ is a limit and we proved the statement for all $\zeta < \xi$. Take A_n, B_n as in the observation and a cofinal ω-type sequence $(\xi_n)_{n\in\omega}$ in ξ. Find paths $P_n \subset A_{2n} \cup B_{2n}$ of order type $\xi_n + k_n$ with $k_n \in \omega$ so that the starting and terminating point of P_n, say u_n, v_n, are in A. Find a finite path Q_n from v_n to u_{n+1} so that $Q_n \setminus \{u_n, v_{n+1}\} \subset A_{2n+1} \cup B_{2n+1}$. The set

$$P_0 \cup Q_0 \cup P_1 \cup Q_1 \cup \ldots$$

is a path of order type ξ. □

Corollary 4.3.3. *If the edges of the complete graph on ω_2 vertices are coloured with countably many colours then we can find a monochromatic path of order type ξ for any $\xi \in \omega_1$.*

Proof. Indeed, we can find a monochromatic copy of $H_{\omega,\omega}$ by Corollary 4.2.2 and hence monochromatic paths of type ξ for any $\xi \in \omega_1$ by Proposition 4.3.1. □

Corollary 4.3.4. *Every graph G with $Col(G) > \omega$ contains a path of order type ξ for all $\xi < \omega_1$.*

Proof. Again, as $H_{\omega,\omega}$ embeds into G if $Col(G) > \omega$, we are done by Proposition 4.3.1. □

[1] $\subseteq^*$ stands for contained modulo finite

Naturally, we would like to see if these results extend to uncountable paths. First, we remark that Observation 1.3.10 and Claim 4.2.1 has the following corollaries on subgraphs of uncountable paths:

Corollary 4.3.5. *Every uncountable path P satisfies $Col(P) > \omega$ and hence P contains a copy of K_{n,ω_1} for all $n \in \omega$ and a copy of $H_{\omega,\omega+1}$.*

Also, it is not hard to see that there are paths which are triangle-free and contain no copies of $H_{\omega,\omega+2}$.

Second, note that $Col(K_{\omega,\omega_1}) = \omega_1$ however K_{ω,ω_1} contains no uncountable paths. Now, we show that not even graphs with uncountable chromatic number necessarily contain uncountable paths. Let us start by an easy observation:

Observation 4.3.6. *Suppose that the tree T does not contain chains of size ω_1. Then the comparability graph $G(T)$ contains no ω_1-unseparable subsets.*

Proof. Take any $A \in [T]^{\omega_1}$ and find incomparable $x, y \in T$. It suffices to check that any finite path from x to y intersects the countable set $x^{\downarrow} \cup y^{\downarrow}$ where

$$t^{\downarrow} = \{s \in T : s \leq t\}.$$

Indeed, the connected component of x in $T \setminus x^{\downarrow}$ contains only elements s with $s \geq x$. □

Corollary 4.3.7. *If T is a non special tree with no uncountable chains then the graph $G = G(T)$ satisfies $Chr(G) > \omega$ while G contains no uncountable ω_1-unseparable subsets.*

Recall that $\sigma\mathbb{Q} = \{s \subseteq \mathbb{Q} : s$ is bounded and well ordered in $\mathbb{Q}\}$ with $s \leq t$ iff s is an initial segment of t is a non special tree with no uncountable chains. Hence

Corollary 4.3.8. *If $G = G(\sigma\mathbb{Q})$ then $|G| = 2^{\omega}$, $Chr(G) = \omega_1$ while there are no uncountable ω_1-unseparable subsets of G; in particular, every path in G is countable.*

Proof. G has uncountable chromatic number by Kurepa's result and the mapping $t \to tp(t) \in \omega_1$ is a good colouring. Thus we are done by Corollary 4.3.7 and Observation 1.3.11. □

Corollary 4.3.7 also gives several consistent examples of G with $|G| = Chr(G) = \omega_1$ without uncountable paths, i.e. if T is a non special Aronszajn tree.

Next, we present a consistent example of a graph G which fails the consequence of Observation 1.3.10 but has uncountable chromatic number; in particular, G contains no uncountable paths. Recall that a Hajnal-Máte graph G on ω_1 is a graph $G = (\omega_1, E)$ of uncountable chromatic number so that $N_G(\alpha) \cap \alpha$ is either finite or a cofinal increasing sequence of type ω in α.

Theorem 4.3.9. *Suppose that $\diamondsuit^*_S$ holds for a stationary, co-stationary $S \subset \omega_1$. Then there is a Hajnal-Máte graph G (so $Chr(G) = |G| = \omega_1$) such that G contains no path of order type ω_1.*

Proof. Recall that $\diamondsuit^*_S$ is the statement that there is sequence $\{\mathcal{C}_\alpha : \alpha \in S\}$ so that $\mathcal{C}_\alpha \in [P(\alpha)]^{\omega}$ and for all $X \subset \omega_1$ there is a club C so that

$$C \cap S \subset \{\alpha \in S : X \cap \alpha \in \mathcal{C}_\alpha\}.$$

Without loss of generality, every $\alpha \in S$ is a limit ordinal.

Now, construct a ladder $C_\alpha \subset \alpha$ for all $\alpha \in S$ so that $A \cap C_\alpha \neq \emptyset$ if $\sup A = \alpha$ and $A \in \mathcal{C}_\alpha$. Our graph is simply $G = (\omega_1, E)$ with

$$E = \{\{\delta, \alpha\} : \delta \in C_\alpha, \alpha \in S\}.$$

It is easy to see now that our graph G cannot contain a path of size ω_1; indeed, $\omega_1 \setminus S$ is stationary and for all $\alpha \in \omega_1 \setminus S$ we have

$$|N_G(\beta) \cap \alpha| < \omega$$

for $\beta \in \omega_1 \setminus \alpha$.

Let us prove now that $Chr(G) > \omega$; it suffices to show that there is no stationary independent subset of S. Indeed, if $T \subset S$ is stationary then, by the definition of $\diamondsuit^*_S$, we can find a club C so that $T \cap \alpha \in \mathcal{C}_\alpha$ for $\alpha \in C \cap S$. Fix any $\alpha \in T' \cap T \cap C$; by the definition of C_α, we must have $C_\alpha \cap T \neq \emptyset$. I. e. if $\delta \in C_\alpha \cap T$ then $\{\delta, \alpha\} \in E$ is an edge in T. □

In contrast, we prove the following

Theorem 4.3.10. *Suppose that MA_κ holds. Then every graph G with $Chr(G) > \omega$ and size $< \kappa$ contains a path of size ω_1.*

As the example $G = K_{\omega,\omega_1}$ shows, the assumption $Col(G) > \omega$ would not be sufficient.

Proof. We fix a graph $G = (V, E)$ of size $< \kappa$ and consider the poset

$$\mathbb{P}_G = \{p \in Fn(V, \omega, \omega) : p \text{ is a good colouring}\}.$$

Recall that $Fn(V, \omega, \omega)$ denotes the set of finite partial functions from V to ω. Now, either $\mathbb{P}_G$ is ccc in which case MA_κ implies that $Chr(G) \leq \omega$ or $\mathbb{P}_G$ contains an uncountable antichain; we show that an uncountable antichain in $\mathbb{P}_G$ implies that G contains an uncountable path.

Fix an antichain $\mathcal{A} = \{p_\xi : \xi < \omega_1\} \subset \mathbb{P}_G$. Let $d_\xi = \operatorname{dom} p_\xi$ and we can suppose that $\{d_\xi : \xi < \omega_1\}$ forms a delta system with kernel d, there is $n \in \omega$ with $|d_\xi| = n$ and there is $r \in Fn(V, \omega, \omega)$ with $p_\xi \restriction d = r$ for all $\xi \in \omega_1$. If there is $\xi \neq \zeta \in \omega_1$ so that there are no edges between $d_\xi \setminus d$ and $d_\zeta \setminus d$ then $p_\xi \cup p_\zeta \in \mathbb{P}_G$. As $\mathcal{A}$ is an antichain, we have that there is an edges between $d_\xi \setminus d$ and $d_\zeta \setminus d$ for any $\xi \neq \zeta \in \omega_1$.

It suffices to prove the following:

Lemma 4.3.11. *Suppose that H is a graph and there is an uncountable family $\mathcal{B} \subset [V(H)]^n$ of disjoint sets for some $n \in \omega \setminus \{0\}$ such that there is an edge between b and b' for all $b \neq b' \in \mathcal{B}$. Then H contains an uncountable path.*

Note that for $n = 1$ the statement is trivial as H contains a complete uncountable graph.

Proof. We actually prove the following slightly stronger statement: if there is $\mathcal{B}_i = \{b_{\xi,i} : \xi \in \omega_1\}$ for $i < 2$ where $b_{\xi,i} \in [V(H)]^n$ pairwise disjoint and there is an edge between $b_{\xi,0}$ and $b_{\zeta,1}$ for all $\xi \leq \zeta < \omega_1$ then H contains an uncountable path.

Consider the following edge colouring c of H_{ω_1,ω_1} (with vertices $\omega_1 \times 2$) with finitely many colours:

$$c((\xi, 0), (\zeta, 1)) = (i, j) \in n^2$$

iff there is an edge between the i^{th} element of $b_{\xi,0}$ and the j^{th} element of $b_{\zeta,1}$ (and (i,j) is the minimal such pair).

It should be clear that a monochromatic path in this edge coloured H_{ω_1,ω_1} gives a path in our original graph H. Indeed, suppose $P = \{p_\nu : \nu < \omega_1\}$ is a path in colour (i,j). For each ν there is $\xi(\nu)$ so that $p_\nu = (\xi(\nu), 0)$ or $p_\nu = (\xi(\nu), 0)$. Let

$$q_\nu = \begin{cases} \text{the } i^{th} \text{ element of } b_{\xi(\nu),0} & \text{if } p_\nu = (\xi(\nu), 0), \\ \text{the } j^{th} \text{ element of } b_{\xi(\nu),1} & \text{if } p_\nu = (\xi(\nu), 1). \end{cases}$$

The sequence $Q = (q_\nu)_{\nu<\omega_1}$ is a path in H.

□

□

We mention the following result of Fremlin [27, Theorem 41H] (which does not imply the existence of uncountable paths):

Theorem 4.3.12. *Suppose that MA_κ holds and the graph $G = (V, E)$ has $Chr(G) > \omega$ and size $< \kappa$. Then*

1. *there is an uncountable $Y \subseteq V$ so that $|N_G[F]| > \omega$ for all $F \in [Y]^{<\omega}$, and*

2. *K_{ω,ω_1} embeds into G.*

Either Theorem 4.3.12(1) or Theorem 4.3.10 implies the following:

Corollary 4.3.13. *If MA_κ holds and G is a graph with uncountable chromatic number and size $< \kappa$ then G contains an uncountable set which is ω_1-unseparable.*

Proof. Apply Theorem 4.3.10 and Observation 1.3.11. □

4.4 Open problems on chromatic number and the subgraph structure

We list some of the intriguing open problems on obligatory subgraphs in this section. A highly popular problem of Erdős and Hajnal (see [17, 18, 24, 49]) is the following:

Problem 4.4.1. *Suppose that $f : \mathbb{N} \to \mathbb{N}$ is increasing. Is it true that there is an uncountably chromatic graph G such that every n-chromatic subgraph of G has at least $f(n)$ vertices (for all $n \geq 3$)?*

P. Komjáth and S. Shelah used forcing and truly virtuoso combinatorial ideas to prove that the answer to the above problem is *consistently yes* [65]. A similar situation is present with the following problem of Erdős and Hajnal (see [17, 21, 22]):

Problem 4.4.2. *Is there an uncountably chromatic graph which contains no triangle free uncountably chromatic subgraphs?*

In [64], Komjáth and Shelah showed that there is a range of forcings which give graphs with the above property and hence the answer is consistently yes.

The finite counterpart of the beautiful Problem 4.4.2 is in fact completely open:

Conjecture 4.4.3 ([16])**.** *For every $k, l \in \mathbb{N}$ there exists $f(k,l) \in \mathbb{N}$ such that any graph with chromatic number at least $f(k,l)$ contains a subgraph of girth bigger than l and chromatic number at least k.*

The only non-trivial advance is due to V. Rödl saying that $f(k,4)$ exists [79].

Let us mention another fascinating conjecture by Erdős:

Conjecture 4.4.4. *Any two uncountably chromatic graphs have a common 4-chromatic subgraph.*

The analogous statement for 3-chromatic subgraphs hold as every two uncountably chromatic graphs contain an odd cycle of the same length by a result of Erdős, Hajnal, Shelah and Thomassen [23, 97].

Chapter 5

The chromatic number and connectivity

Observe that any graph G with uncountable chromatic number contains a connected component H with uncountable chromatic number. Furthermore, a graph G with uncountable chromatic number always contains a copy of $K_{n,n}$ and hence an n-connected subgraph for each $n \in \mathbb{N}$. How much more can we say about connected subgraphs of graphs with large chromatic number?

5.1 An overview of previous results

Erdős and Hajnal [20] asked if one can find subgraphs of large chromatic number and high connectivity in graphs with large chromatic number. In particular, in [20] they ask if every graph with chromatic number and size ω_1 contains a subgraph of chromatic number ω_1 which is ω-connected i.e. any two points are connected by infinitely many pairwise disjoint paths. In 1985, Erdős and Hajnal asked if every graph of chromatic number ω_1 contains an uncountably chromatic ω-connected subgraph [22]; these problems are included in the recent surveys [61] and [63] as well.

Most advances on these questions are due to Péter Komjáth. First, he proved

Theorem 5.3.1 ([55]). *Every uncountably chromatic graph G contains n-connected uncountably chromatic subgraphs H_n for every $n \in \mathbb{N}$.*

The goal of this chapter is to provide a rather simple proof to this theorem in Section 5.3 by modern methods, with the use of Davies-trees. We remark that Komjáth also proves in [55] that one can find such subgraphs H_n which have minimal degree ω; we were not able to deduce this stronger result with our tools.

Regarding infinitely connected subgraphs, he shows the following

Theorem 5.1.1 ([57]). *Suppose $V \models CH$ and G is a graph of size ω_1 that has no uncountably chromatic ω-connected subgraphs. Then there is a ccc forcing $\mathbb{P}_G$ of size ω_1 such that $V^{\mathbb{P}_G} \models Chr(G) \leq \omega$.*

In particular, under PFA (the Proper Forcing Axiom) every graph of size and chromatic number ω_1 contains an uncountably chromatic ω-connected subgraph; this is a simple application of the cardinal collapsing trick [6]. On the other hand

Theorem 5.1.2 ([57]). *It is consistent that there is a graph G of size and chromatic number ω_1 such that every ω-connected subgraph of G is countably chromatic.*

Hence the original question of Erdős and Hajnal from [20] is independent of ZFC. Komjáth's proof was slightly flawed (see the introduction of [62] for details), but the error is corrected in the recent [62] where he forces a stronger example:

Theorem 5.1.3 ([62]). *It is consistent that there is a graph of size and chromatic number ω_1 without* uncountable *ω-connected subgraphs.*

Note that the answer to the question from [22] is consistently no. In Chapter 6, we show that there are graphs with size 2^ω and chromatic number ω_1 without uncountable ω-connected subgraphs purely in ZFC and hence we provide a complete solution to the question of Erdős and Hajnal from [22].

5.2 Davies-trees and infinite combinatorics

The goal of this section is to introduce the notion of *Davies-trees* and review applications prior to our work.

5.2.1 An introduction to Davies-trees

Let us define and prove the existence of *Davies-trees* at the same time:

Fact 5.2.1. *Suppose that $\mathcal{A}$ is a countable set and $\mathcal{X}$ is an arbitrary set. Then there is a large enough cardinal Θ and a sequence of $\mathcal{M} = (M_\alpha)_{\alpha<\kappa}$ of countable elementary submodels of $H(\Theta)$ so that*

1. *$\{\mathcal{X}\} \cup \mathcal{A} \subset M_\alpha$ for all $\alpha < \kappa$,*
2. *$\mathcal{X} \subset \bigcup_{\alpha<\kappa} M_\alpha$,*
3. *for every $\beta < \kappa$ there is $m_\beta \in \mathbb{N}$ and models $N_{\beta,i} \prec H(\Theta)$ such that $\{\mathcal{X}\} \cup \mathcal{A} \subset N_{\beta,i}$ for $i < m_\beta$ and*
$$\bigcup\{M_\alpha : \alpha < \beta\} = \bigcup\{N_{\beta,i} : i < m_\beta\}.$$

We will refer to a sequence of models $\mathcal{M}$ with property (3) as a *Davies-tree.*

Note that if the sequence $(M_\alpha)_{\alpha<\kappa}$ is increasing then $\bigcup\{M_\alpha : \alpha < \beta\}$ is also an elementary submodel of $H(\Theta)$ for each $\beta < \kappa$; however, there is no way to cover a set of size bigger than ω_1 with an increasing chain of *countable* sets. Fact 5.2.1 says that we can cover by countable elementary submodels and almost maintain the property that the initial segments $\bigcup\{M_\alpha : \alpha < \beta\}$ are submodels. Indeed, each initial segment is the union of *finitely many* submodels by condition (3) while these models contain everything relevant (denoted by $\mathcal{A}$ above) as well.

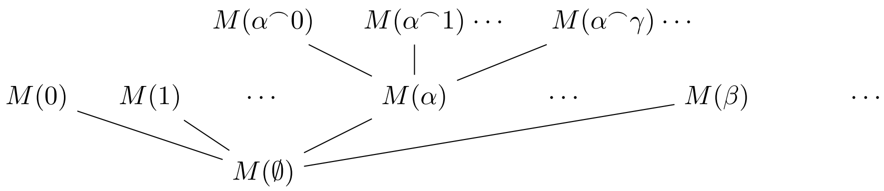
Proof. Suppose that $\mathcal{X}$ has size λ. We recursively construct a tree T of finite sequences of ordinals and elementary submodels $M(a)$ for $a \in T$. Let $\emptyset \in T$ and let $M(\emptyset)$ be an elementary submodel of size λ so that

- $\{\mathcal{X}\} \cup \mathcal{A} \subset M(\emptyset)$,
- $\mathcal{X} \subset M(\emptyset)$.

Suppose that we defined a tree T' and corresponding models $M(a)$ for $a \in T'$. Fix $a \in T'$ and suppose that $M(a)$ is *uncountable*. Find a continuous sequence of elementary submodels $(M(a^\frown\xi))_{\xi<\zeta}$ so that

- $\{\mathcal{X}\} \cup \mathcal{A} \subset M(a^\frown\xi)$ for all $\xi < \zeta$,
- $M(a^\frown\xi)$ has size less than $M(a)$.

We extend T' with $\{a^\frown\xi : \xi < \zeta\}$ and iterate this procedure to get T.



It is easy to see that this process produces a downwards closed subtree T of $Ord^{<\omega}$ and if $a \in T$ is a branch then $M(a)$ is countable. Let us well order $\{M(a) : a \in T \text{ is a branch}\}$ by the lexicographical ordering.

We wish to show that if $b \in T$ is a branch then $\bigcup\{M(a) : a <_{lex} b, a \in T \text{ is a branch}\}$ is the union of finitely many submodels containing $\{\mathcal{X}\} \cup \mathcal{A}$. Suppose that $|b| = n \in \mathbb{N}$ and write

$$N_{b,i} = \bigcup\{M((b \restriction i-1)^\frown\xi) : \xi < b(i-1)\}$$

for $i = 1 \ldots n$. It is clear that $N_{b,i}$ is an elementary submodel as a union of an increasing chain. Also, if $a <_{lex} b$ then $M(a) \subset N_{b,i}$ must hold where $i = \min\{j \leq n : a(j) \neq b(j)\}$.

□

Remark: note that this proof shows that if $\mathcal{X}$ has size $\aleph_n$ then every initial segment in the lexicographical ordering is the union of n elementary submodels (the tree T has height n).

In the future, when working with a sequence of elementary submodels $\mathcal{M} = (M_\alpha)_{\alpha<\kappa}$, we use the notation

$$\mathcal{M}_{<\beta} = \bigcup\{M_\alpha : \alpha < \beta\}$$

for $\beta < \kappa$.

5.2.2 The first applications

The very first

As we mentioned already, the above constructed tree of models is originated in the work of Roy O. Davies [9] from the early 60's. He proves that the plane $\mathbb{R}^2$ can be covered by countably many rotated graphs of functions; this was known to be true under the Continuum Hypothesis (proved by Sierpinski in the 30's) while Davies' result holds regardless of cardinal arithmetic.

The importance of the tree construction is that we can cover arbitrary large structures (in this case $\mathbb{R}^2$) with countable sets in a way that initial segments are fairly close to models (unions of finitely many models). This way the assumption of CH can be eliminated from Sierpinski's original result.

The Steinhaus tiling problem

Probably the most important application of Davies-trees is S. Jackson and R. D. Mauldin's solution from 2002 to the Steinhaus tiling problem. In the late 50's H. Steinhaus asked if there is a subset S of $\mathbb{R}^2$ such that every rotation of S tiles the plane or equivalently, S intersects every isometric copy of the lattice $\mathbb{Z} \times \mathbb{Z}$ in exactly one point. Jackson and Mauldin provides an affirmative answer (surveyed in [48]); their proof elegantly combines hard combinatorial, geometrical and set theoretical methods (a transfinite induction using Davies-trees).

Again, their proof becomes somewhat simpler assuming CH. However, this assumption can be eliminated, as before, if one uses Davies-trees as a substitute for increasing chains of models.

Topology

In 2008, D. Milovich published a paper [71] in set theoretic topology (order theory of bases) where he further polished Jackson and Mauldin's Davies-tree decomposition technique. In particular, one can guarantee that the Davies-tree $(M_\alpha)_{\alpha<\kappa}$ has the additional property that $\{N_{\alpha,i} : i < m_\alpha\} \in M_\alpha$ for all $\alpha < \kappa$. This extra hypothesis is very useful in several situations.

One can find easy-to-read introduction to Davies-trees in the presentations of D. Milovich [70, 72] (with slightly different terminology) and L. Soukup [95]. Also, it is likely that there are other papers, even earlier then Davies', where similar techniques appear either explicitly or implicitly however at the point of writing this note we are not aware of further references.

5.3 The chromatic number and n-connected subgraphs

We proceed with our main application of Davies-trees which highly simplifies the original proof.

Theorem 5.3.1. *Every uncountably chromatic graph G contains n-connected uncountably chromatic subgraphs for every $n \in \mathbb{N}$.*

Fix a graph $G = (V, E)$, $n \in \omega$ and consider the set $\mathcal{A}$ of all subsets of V spanning *maximal n-connected subgraphs* of G.

We will follow Komjáth's framework in the sense that we are going to define a *good ordering* on $\mathcal{A}$. The following lemma explains what we mean by a good ordering.

Lemma 5.3.2. *Suppose that $G = (V, E)$ is a graph, $\{A_\xi : \xi < \mu\}$ is a cover of V with countably chromatic subsets so that $|N_G(x) \cap \bigcup A_{<\xi}| < \omega$ for all $\xi < \mu$ and $x \in A_\xi \setminus \bigcup A_{<\xi}$ where $A_{<\xi} = \{A_\zeta : \zeta < \xi\}$. Then $Chr(G) \leq \omega$.*

Proof. Suppose that $g_\xi : A_\xi \to \omega$ witnesses that the chromatic number of A_ξ is $\leq \omega$. We define $f : V \to \omega \times \omega$ by defining $f \restriction (A_\xi \setminus \bigcup A_{<\xi})$ by induction on $\xi < \mu$. If $x \in A_\xi \setminus \bigcup A_{<\xi}$ then the first coordinate of $f(x)$ is $g_\xi(x)$ while the second coordinate of $f(x)$ avoids all the finitely many second coordinates appearing in $\{f(y) : y \in N_G(x) \cap \bigcup A_{<\xi}\}$. It is easy to see that f witnesses that G has countable chromatic number. $\square$

Let us continue with some straightforward observations about the maximal n-connected sets:

Observation 5.3.3. *1. $A \not\subseteq A'$ for all $A \neq A' \in \mathcal{A}$,*

2. $|A \cap A'| < n$ *for all* $A \neq A' \in \mathcal{A}$,

3. $|\{A \in \mathcal{A} : a \subset A\}| \leq 1$ *for all* $a \in [V]^{\geq n}$,

4. $|N_G(x) \cap A| < n$ *for all* $x \in V \setminus A$ *and* $A \in \mathcal{A}$.

The next claim is fairly simple and describes a situation when we can join n-connected sets.

Claim 5.3.4. *Suppose that* $A_i \subset V$ *spans an* n*-connected subset for each* $i < n$ *and we can find* $Y = \{y_{i,k} : i < n, k < n\}$ *and* $X = \{x_k : k < n\}$ *distinct points so that*

$$y_{i,k} \in A_i \cap N_G(x_k)$$

for all $i < n, k < n$. *Then* $A = \bigcup\{A_i : i < n\} \cup X$ *is* n*-connected.*

Proof. Let $F \in [A]^{<r}$ and note that there is a $k < n$ so that $\{y_{i,k}, x_k : i < n\} \cap F = \emptyset$ for some $k < n$. Thus $\cup\{A_i : i < n\} \cup \{y_{i,k}, x_k : i < n\} \setminus F$ is connected as $A_i \setminus F$ is connected for all $i < n$. Finally, if $x_j \in A \setminus F$ then $N_G(x_j) \cap \cup\{A_i : i < n\} \setminus F \neq \emptyset$ so we are done. ☐

Now, we deduce some useful facts about elementary submodels and maximal n-connected sets.

Lemma 5.3.5. *Suppose that* $N \prec H(\Theta)$ *with* $G \in N$ *and*

$$|N_G(x) \cap N| \geq n$$

for some $x \in V \setminus N$. *Then* $x \in A$ *for some* $A \in \mathcal{A} \cap N$.

Proof. Let $a \in [N_G(x) \cap N]^n$. There is a copy of K_{n,ω_1} (complete bipartite graph with classes of size n and ω_1) which contains $a \cup \{x\}$ (see Claim 4.2.1). As K_{n,ω_1} is n-connected, there must be $A \in \mathcal{A}$ with $a \cup \{x\} \subset A$ as well. Also, there is $A' \in \mathcal{A} \cap N$ with $a \subset A'$ by elementarity; as $|A \cap A'| \geq n$ we have $A = A'$ which finishes the proof. ☐

Lemma 5.3.6. *Suppose that* $N \prec H(\Theta)$ *with* $G \in N$ *and*

$$|N_G(x) \cap \bigcup(\mathcal{A} \cap N)| \geq \omega$$

for some $x \in V \setminus N$. *Then* $x \in A$ *for some* $A \in \mathcal{A} \cap N$.

Proof. Suppose that the conclusion fails; by the previous lemma, we have $|N_G(x) \cap N| < n$. In particular, there is sequence of distinct $A_i \in \mathcal{A} \cap N$ for $i < n$ so

$$(N_G(x) \cap A_i) \setminus N \neq \emptyset$$

for all $i < n$ (as $N_G(x) \cap A$ is finite if $A \in N \cap \mathcal{A}$).

Thus

$$N \models \forall F \in [V]^{<\omega} \exists x \in V \setminus F \text{ and } y_i \in (A_i \cap N_G(x)) \setminus F.$$

Now, we can find distinct $\{y_{i,k} : i < n, k < n\}$ and $X = \{x_k : k < n\}$ so that

$$y_{i,k} \in A_i \cap N_G(x_k).$$

Finally, $\cup\{A_i : i < n\} \cup X$ is n-connected by Claim 5.3.4 which contradicts the maximality of A_i. ☐

Proof of Theorem 5.3.1. Let $G, \mathcal{A}$ be as above and suppose that every $A \in \mathcal{A}$ is countably chromatic; we will show that in this case, G is countably chromatic.

First, we prove that $\bigcup \mathcal{A}$ is countably chromatic. Take a Davies-tree covering $\mathcal{A}$ i.e. a sequence $(M_\alpha)_{\alpha<\kappa}$ of countable elementary submodels such that for all $\alpha < \kappa$ there is a *finite* sequence of elementary submodels $(N_{\alpha,j})_{j<m_\alpha}$ so that

$$\bigcup \mathcal{M}_{<\alpha} = \bigcup \{N_{\alpha,j} : j < m_\alpha\},$$

with $G \in M_\alpha \cap N_{\alpha,j}$ and $\mathcal{A} \subset \bigcup\{M_\alpha : \alpha < \kappa\}$.

Let $\mathcal{A}_{<\alpha} = \mathcal{A} \cap \bigcup \mathcal{M}_{<\alpha}$ and $\mathcal{A}_\alpha = (\mathcal{A} \cap M_\alpha) \setminus \mathcal{A}_{<\alpha}$ for $\alpha < \kappa$. Well order $\mathcal{A}$ as $\{A_\xi : \xi < \mu\}$ so that

1. $A_\zeta \in \mathcal{A}_{<\alpha}, A_\xi \in \mathcal{A} \setminus \mathcal{A}_{<\alpha}$ implies $\zeta < \xi$ and
2. $\mathcal{A}_\alpha \setminus \mathcal{A}_{<\alpha}$ has order type $\leq \omega$

for all $\alpha < \kappa$.

We claim that the above enumeration of $\mathcal{A}$ satisfies Lemma 5.3.2 and thus $\bigcup \mathcal{A}$ is countably chromatic. By the second property of our enumeration and Observation 5.3.3 (4), it suffices to show that

$$|N_G(x) \cap \bigcup \mathcal{A}_{<\alpha}| < \omega$$

if $x \in A \setminus \bigcup \mathcal{A}_{<\alpha}$ for all $A \in \mathcal{A}_\alpha \setminus \mathcal{A}_{<\alpha}$ and $\alpha < \kappa$.

However, as $\mathcal{A}_{<\alpha} = \bigcup\{\mathcal{A} \cap N_{\alpha,j} : j < m_\alpha\}$, this should be clear from applying Lemma 5.3.6 for each of the finitely many models $N_{\alpha,j}$ where $j < m_\alpha$.

Now, we show that G is countably chromatic; otherwise, the graph spanned by $V \setminus \bigcup \mathcal{A}$ is uncountably chromatic. However, every uncountably chromatic graph, and so $V \setminus \bigcup \mathcal{A}$ as well, contains an n-connected subgraph (actually a copy of K_{n,ω_1}) which contradicts the definition of $\mathcal{A}$. $\square$

5.4 Further applications of Davies trees

Degrees of disjointness

We start by proving a simple fact from the theory of almost disjoint set systems.

Definition 5.4.1. *We say that a family of sets $\mathcal{X}$ is* n-almost disjoint *for some $n \in \mathbb{N}$ iff $|A \cap B| < n$ for every $A \neq B \in \mathcal{X}$. $\mathcal{X}$ is* essentially disjoint *iff we can select finite $F_A \subset A$ for each $A \in \mathcal{A}$ so that $\{A \setminus F_A : A \in \mathcal{A}\}$ is disjoint.*

Theorem 5.4.2 ([53])**.** *Every n-almost disjoint family $\mathcal{X}$ of countable sets is essentially disjoint for every $n \in \mathbb{N}$.*

Proof. Take a Davies-tree $\mathcal{M} = \{M_\alpha : \alpha < \kappa\}$ such that $\mathcal{X} \subset \bigcup \mathcal{M}$ and that $\mathcal{X} \in M_\alpha$ for each $\alpha < \kappa$. Recall that $\bigcup \mathcal{M}_{<\alpha} = \bigcup\{N_{\alpha,i} : i < m_\alpha\}$ for each $\alpha < \kappa$. We define a map F on $\mathcal{X}$ such that $F(A) \in [A]^{<\omega}$ for each $A \in \mathcal{X}$ and $\{A \setminus F(A) : A \in \mathcal{X}\}$ is pairwise disjoint.

Let $\mathcal{X}_\alpha = (\mathcal{X} \cap M_\alpha) \setminus \bigcup \mathcal{M}_{<\alpha}$ and $\mathcal{X}_{<\alpha} = \mathcal{X} \cap (\bigcup \mathcal{M}_{<\alpha})$. We define F on each $\mathcal{X}_\alpha$ independently so fix $\alpha < \kappa$.

Observation 5.4.3. $|A \cap (\bigcup \mathcal{X}_{<\alpha})| < \omega$ *for all $A \in \mathcal{X}_\alpha$.*

Proof. Otherwise, there is $i < m_\alpha$ so that $A \cap \bigcup(\mathcal{X} \cap N_{\alpha,i})$ is infinite and in particular, we can select $a \in [A \cap \bigcup(\mathcal{X} \cap N_{\alpha,i})]^n$. Note that $\bigcup(\mathcal{X} \cap N_{\alpha,i}) \subset N_{\alpha,i}$ as each set in $\mathcal{X}$ is countable hence $a \subset N_{\alpha,i}$ and $a \in N_{\alpha,i}$. However, $N_{\alpha,i} \models$ "there is a *unique* element of $\mathcal{X}$ containing a" (by n-almost disjointness) hence $A \in N_{\alpha,i} \subset \bigcup \mathcal{M}_{<\alpha}$ (by elementarity) which contradicts $A \in \mathcal{X}_\alpha$. □

Now list $\mathcal{X}_\alpha$ as $\{A_{\alpha,l} : l \in \omega\}$. Let

$$F(A_{\alpha,l}) = A_{\alpha,l} \cap \left(\bigcup \mathcal{X}_{<\alpha} \cup \bigcup \{A_{\alpha,k} : k < l\}\right)$$

for $l < \omega$. Clearly, F witnesses that $\mathcal{X}$ is essentially disjoint.

□

Clouds above the Continuum Hypothesis

The next theorem we prove has a certain similarity to Davies' result. The reason that this proof is of greater interest is that it highlights the fact that a set of size $\aleph_n$ can be covered by a Davies-tree such that the initial segments are expressed as the union of n elementary submodels (for $n \in \mathbb{N}$). The same fact is utilized in an application presented in [95].

Definition 5.4.4. *We say that $A \subset \mathbb{R}^2$ is a* cloud around a point $a \in \mathbb{R}^2$ *iff every line l through a intersects A in a finite set.*

Note that one or two clouds cannot cover the plane; indeed, if A_i is a cloud around a_i for $i < 2$ then the line l through a_0 and a_1 intersects $A_0 \cup A_1$ in a finite set. How about three or more clouds?

Theorem 5.4.5 ([60] and [85])**.** *The following are equivalent for each $n \in \mathbb{N}$:*

1. $2^\omega \leq \aleph_n$,
2. $\mathbb{R}^2$ *is covered by at most $n + 2$ clouds.*

We only prove (1) implies (2) and follow Komjáth's original proof for the $2^\omega = \omega_1$ case.

Proof. Fix $n \in \omega$ and suppose that the continuum is $\aleph_n$. This implies that $\mathbb{R}^2$ can be covered by a Davies-tree $\{M_\alpha : \alpha < \kappa\}$ so that $\bigcup \mathcal{M}_{<\alpha} = \bigcup\{N_{\alpha,i} : i < n\}$ for every $\alpha < \kappa$.

Fix $n + 2$ points $\{a_k : k < n + 2\}$ in $\mathbb{R}^2$ in general position (i.e. no three are collinear). Let $\mathcal{L}^k$ denote the set of lines through a_k and let $\mathcal{L} = \bigcup\{\mathcal{L}^k : k < n + 2\}$. We will define clouds A_k around a_k by defining a map $F : \mathcal{L} \to [\mathbb{R}^2]^{<\omega}$ such that $F(l) \in [l]^{<\omega}$ and letting

$$A_k = \{a_k\} \cup \bigcup\{F(l) : l \in \mathcal{L}^k\}$$

for $k < n + 2$. We have to make sure that for every $x \in \mathbb{R}^2$ there is $l \in \mathcal{L}$ so that $x \in F(l)$.

Now let $\mathcal{L}_\alpha = (\mathcal{L} \cap M_\alpha) \setminus \bigcup \mathcal{M}_{<\alpha}$ and $\mathcal{L}_{<\alpha} = \mathcal{L} \cap \bigcup \mathcal{M}_{<\alpha}$ for $\alpha < \kappa$. We define F on L_α for each $\alpha < \kappa$ independently.

Fix an $\alpha < \kappa$ and list $\mathcal{L}_\alpha \setminus \mathcal{L}'$ as $\{l_{\alpha,j} : j < \omega\}$ where $\mathcal{L}'$ is the set of $\binom{n+2}{2}$ lines determined by $\{a_k : k < n + 2\}$. We let

$$F(l_{\alpha,j}) = \bigcup\{l \cap l_{\alpha,j} : l \in \mathcal{L}' \cup \{l_{\alpha,j'} : j' < j\}\}$$

for $j < \omega$.

We claim that this definition works: fix a point $x \in \mathbb{R}^2$ and we will show that there is $l \in \mathcal{L}$ with $x \in F(l)$. Find the unique $\alpha < \kappa$ such that $x \in M_\alpha \setminus \bigcup \mathcal{M}_{<\alpha}$. It is easy to see that $\cup\mathcal{L}'$ is covered by our clouds hence we suppose $x \notin \bigcup \mathcal{L}'$. Let l_k denote the line through x and a_k.

Observation 5.4.6. *$|\bigcup \mathcal{M}_{<\alpha} \cap \{l_k : k < n+2\}| \leq n$.*

Proof. Suppose that this is not true. Then (by the pigeon hole principle) there is $i < n$ such that $|N_{\alpha,i} \cap \{l_k : k < n+2\}| \geq 2$ and in particular the intersection of any two of these lines, the point x, is in $N_{\alpha,i} \subset \bigcup \mathcal{M}_{<\alpha}$. This contradicts the choice of α. □

We have now that

$$|\{l_k : k < n+2\} \cap (\mathcal{L}_\alpha \setminus \mathcal{L}')| \geq 2$$

i.e. there is $j' < j < \omega$ such that $l_{\alpha,j'}, l_{\alpha,j} \in \{l_k : k < n+2\}$. Hence $x \in F(l_{\alpha,j})$ is covered by one of the clouds.

□

Conflict free chromatic number

Let us mention, without proof, the following application of Davies trees due to L. Soukup [95].

Definition 5.4.7. *If $\mathcal{A}$ is a set system then the* conflict free chromatic number of $\mathcal{A}$ *is the least cardinal κ so that there is a map $f : \mathcal{A} \to \kappa$ such that for every $A \in \mathcal{A}$ there is $\alpha < \kappa$ with $|A \cap f^{-1}(\alpha)| = 1$.*

Theorem 5.4.8 ([41])**.** *Let $m, d \in \omega$ and suppose that $\mathcal{A} \subseteq [\omega_m]^\omega$ is d-almost disjoint. Then the conflict free chromatic number of $\mathcal{A}$ is at most $\lfloor \frac{(m+1)(d-1)+1}{2} \rfloor + 2$*

Future work

There seems to be great possibilities in the use of Davies-trees beyond finding new proofs or eliminating CH from known results. Recently, L. Soukup started to develop the analogue of Davies-trees with σ-closed models.

Theorem 5.4.9 ([95])**.** *Suppose $V = L$. Then for every cardinal κ there is a sequence $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of $H(\Theta)$ covering κ such that*

1. *$[M_\beta]^\omega \subset M_\beta$ and $|M_\beta| = \omega_1$,*
2. *there are $N_{\beta,j} \prec H(\Theta)$ with $[N_{\beta,j}]^\omega \subset N_{\beta,j}$ for $j < \omega$ such that*

$$\bigcup\{M_\alpha : \alpha < \beta\} = \bigcup\{N_{\beta,j} : j < \omega\}$$

for all $\beta < \kappa$.

The presentation [95] contains further details on σ-Davies-trees. In particular, L. Soukup provides a new proof of the fact that $V = L$ implies that the poset $\langle [\kappa]^\omega, \subseteq \rangle$ has the *weak Freese-Nation property* [28] i.e. there is a map $f : [\kappa]^\omega \to [[\kappa]^\omega]^\omega$ so that $p \leq q$ implies that $p \leq r \leq q$ for some $r \in f(p) \cap f(q)$ for all $p, q \in [\kappa]^\omega$.

Chapter 6

The chromatic number and infinitely connected subgraphs

The main goal of this chapter is to construct a graph of chromatic number ω_1 without an uncountable infinitely connected subgraph. This will be done by defining graphs through ladder systems on non special trees. After a short introduction, we present the main construction in Section 6.3 followed by further results, general remarks and open problems.

6.1 A short introduction to trees and ladder systems

D. Kurepa was the first to systematically study set theoretic trees [68]; his work on Souslin's Problem lead to several fundamental results on Aronszajn and Souslin trees. We refer the reader to the survey [47] for further details on Kurepa's work. In the Handbook of Set Theoretic Topology, S. Todorcevic [99] gives an excellent review of the most important results and references on the topic of trees and linearly ordered sets up to 1984. More recently, the survey by J. Moore [75] on Aronszajn trees summarizes the advances made since the publication of [99].

Trees with strong combinatorial properties played an important part in solving deep problems of set theoretical nature, in particular in general topology and the theory of partition relations. Let us mention that M. E. Rudin used Souslin trees to construct S-spaces [80] and Dowker spaces [81]. S. Todorcevic extended the theory of partition relations from ordinals to partially ordered sets through trees [100] and he uses special Aronszajn trees to construct square bracket colourings on ω_1 in ZFC [101]. On a related note, Shelah used a Souslin tree to aid the construction of a square bracket colouring with no rainbow triangles in [86].

Undoubtedly, our most important reference for this chapter is Todorcevic's inspirational "Stationary sets, trees and continuums" [98]. We already recalled the definition of the trees $T(S)$ (where $S \subseteq \omega_1$) and listed their most important properties in Section 1.5.

We are interested in graphs associated to trees through the comparability relation i.e. for a tree $(T, \leq_T)$ we look at the graph $G(T) = (T, E)$ where $\{s, t\} \in E$ iff $s \leq_T t$. The class of graphs which arise as comparability graphs of some partial order were studied in detail [103, 104, 32, 31, 33]. While every finite comparability graph G is perfect [73] i.e. $Chr(G)$ is the size of the largest clique, this theorem does not extend to the infinite case (see [84] for an exposition). Indeed, we will exploit the fact that for the above defined tree $T = T(S)$, when S is stationary and costationary, the comparability graph $G(T)$ has chromatic number ω_1 while $G(T)$ has no uncountable cliques (as T has no uncountable branches).

Ladder systems on the ordinal ω_1 played a central role in recent advances in set theory. We mention two outstanding examples: the ground breaking work of S. Todorcevic on minimal walks [101] and Shelah's work on uniformization and his solution to the Whitehead problem [88, 89].

Graphs G on ω_1 with chromatic number ω_1 where the edge relation is given by a ladder system were introduced in [45] and hence named Hajnal-Máté graphs. In [45], the authors prove that MA_{ω_1} implies that there are no Hajnal-Máté graphs while one can construct Hajnal-Máté graphs under $\diamondsuit^*$. Let us remark that Abraham, Devlin and Shelah proved that CH is not sufficient for the existence of Hajnal-Máté graphs [1]; we believe that it is not known whether $\clubsuit$ implies the existence of Hajnal-Máté graphs.

P. Komjáth has a series of papers on the subject exploring several constructions with further interesting properties: in [52], he constructs Hajnal-Máté graphs from $\diamondsuit$ which contain no triangles; in [54], he constructs Hajnal-Máté graphs from $\diamondsuit^*$ that contain no cycles which are the union of two $<_{\omega_1}$-monotone paths; in [58], the previous $\diamondsuit^*$ construction is further developed to have certain extra structural properties. Later, Komjáth and Shelah [64] showed that one can even exclude $K_{\omega,\omega}$ (even $H_{\omega,\omega+2}$) and odd cycles up to a given length at the same time using Hajnal-Máté graphs; this is done by forcing and $\diamondsuit$ constructions as well. Finally, in [64, Theorem 10], the authors refer to an idea of F. Galvin and carry out the Hajnal-Máté graph construction based on the tree $\bigcup\{\omega^\alpha : \alpha < \omega_1\}$ purely in ZFC[1]. Recently, U. Abraham and Y. Yin [2] investigated the chromatic number of a related class of graphs.

Our aim in this chapter is to use ladder systems not on ω_1 but on non special trees T in order to select a Hajnal-Máté-like nice subgraph of the comparability graph $G(T)$ in ZFC, without the use of any guessing principles. We are only aware of a handful of references where trees and ladder systems appear in such close relation. The most relevant reference is the above mentioned [64, Theorem 10]. The paper [96] by Z. Spasojević deals with the uniformization properties of ladder systems on trees. The notion of a T-uniformization of ladder systems on ω_1 for a tree T appears in Moore's [74].

Our proofs in this chapter were deeply motivated by arguments in [98], as well as by certain diagonalization arguments of M. E. Rudin [82] and Z. Balogh [5] using elementary submodels.

6.2 Preliminaries

We will use the following terminology:

Definition 6.2.1. *We say that a set of vertices F in a graph* separates *two vertices s and t iff every path from s to t passes through F. We say that F separates a set of vertices A iff there are distinct $s, t \in A$ such that F separates s and t.*

Hence a graph G is ω-connected iff no finite set separates two points of G. Furthermore, recall

Corollary 4.3.7. *If T is a non special tree with no uncountable chains then the graph $G = G(T)$ satisfies $Chr(G) > \omega$ while every uncountable set of vertices can be separated by a countable set.*

The above simple corollary indicates that it is reasonable to investigate non special trees T without uncountable chains and the corresponding graphs $G(T)$ regarding the Erdős-Hajnal question.

[1] We were not aware of this reference when proving the results of Chapter 6.

Indeed, we will work with trees of the form $T(S) = \{t \subset S : t \text{ is closed}\}$ where $S \subseteq \omega_1$ is stationary, costationary. Recall from Section 1.5 that the comparability graph of $T(S)$ has size 2^ω and chromatic number ω_1.

Our aim is to construct a subgraph X of the comparability graph of $T(S)$ (where $S \subseteq \omega_1$ is stationary, costationary) which has chromatic number ω_1 and does not contain uncountable *ω-connected* subsets i.e. every uncountable set of vertices A contains two points $s, t \in A$ and a finite set $F \subset A$ such that any path $P \subset A$ between s and t passes through F.

6.3 The main construction

We start by defining *ladder system graphs* on trees and characterizing connectivity properties by simple combinatorial facts about the ladder system.

Definition 6.3.1. *Suppose that T is a tree. A* ladder system on T *is a family $\underline{C} = \{C_t : t \in T\}$ so that $C_t \subset t^\downarrow$ is either finite or a cofinal sequence of type ω.*

Each ladder system $\underline{C}$ defines a subgraph $X_{\underline{C}}$ of $G(T)$ with vertices T and edges

$$\{\{s,t\} : s \in C_t, t \in T\}.$$

This is in direct analogy with the Hajnal-Máté graphs introduced in [45] i.e. the case where the tree T is simply the cardinal ω_1.

Definition 6.3.2. *A ladder system $\underline{C}$ on T is* transitive *iff*

$$C_t \cap s^\downarrow \subseteq C_s$$

for all $t \in T$ and $s \in C_t$.

Note that $\underline{C}$ is transitive iff C_t spans a complete graph in $X_{\underline{C}}$ for all $t \in T$. The next two lemmas explain why we introduced the notion of a transitive ladder system.

Lemma 6.3.3. *Suppose that T is a tree and $\underline{C}$ is a transitive ladder system on T. If $s, t \in T$ and P is a finite path in $X_{\underline{C}}$ from s to t then there is a path $Q \subseteq P$ which is the union of two monotone paths.*

A *monotone path* in $X_{\underline{C}}$ is a path which is a chain in the tree ordering.

Proof. Let $Q \subseteq P$ be a path of *minimal length* from s to t. Let $\{q_i : i < n\}$ enumerate Q by its path ordering. Note that we cannot have $q_{i-1}, q_{i+1} < q_i$ for any $1 \leq i \leq n-1$; indeed, this would imply that $q_{i-1}, q_{i+1} \in C_{q_i}$ and hence, by transitivity, q_{i-1} and q_{i+1} are connected by an edge which contradicts the minimality of Q. This means that if $q_{i-1} < q_i$ then $q_i < q_{i+1}$ for any $1 \leq i \leq n-1$ i.e. Q is monotone increasing from the first step up (in the tree ordering). In other words, Q is the union of a monotone decreasing and a monotone increasing path.

□

Lemma 6.3.4. *Suppose that T is a tree and $\underline{C}$ is a transitive ladder system on T. If T has no branching at limit levels and contains no uncountable chains then $X_{\underline{C}}$ contains no uncountable ω-connected subsets.*

Proof. Fix an uncountable $A \subseteq T$ and find two incomparable elements $s, t \in A$. Let r denote the maximal common initial part of s and t; this exists and $r < s, t$ as T does not branch at limits. Find $s' \in A$ so that $r < s' \leq s$ and

$$s'^{\downarrow} \cap A \subseteq r^{\downarrow} \cup \{r\}.$$

We claim that s' and t are separated by the finite set $F = \{r\} \cup (r^{\downarrow} \cap C_{s'})$.

Suppose that $P = \{p_i : i < n\} \subset A$ is a finite path from $p_0 = s'$ to t. By Lemma 6.3.3, we can suppose that P is the union of two monotone paths. Note that $p_1 \in A \cap C_{s'}$ as $p_1 < p_0 = s'$ and hence $p_1 \in r^{\downarrow} \cup \{r\}$ by the choice of s'. In particular, $p_1 \in F$; this shows that F separates s' and t. □

We are ready now to prove our main result:

Theorem 6.3.5. *Fix a stationary, costationary $S \subset \omega_1$ and let $T = T(S)$. Then there is a subgraph X of $G(T)$ such that $Chr(X) = \omega_1$ and X contains no uncountable ω-connected subsets.*

We use the following notation: if $T = T(S)$ for some $S \subseteq \omega_1$ then let

$$T_\delta = \{t \in T : \max(t) = \delta\}$$

for $\delta \in \omega_1$ and similarly define $T_{<\delta}, T_{\leq\delta}$.

Proof. It suffices to show that there is a transitive ladder system $\underline{C}$ on T such that $Chr(X_{\underline{C}}) = \omega_1$; indeed, T has no uncountable chains nor branches at limit levels hence Lemma 6.3.4 implies that $X_{\underline{C}}$ has no uncountable ω-connected subsets. Furthermore, as the tree T has height ω_1 we have that $Chr(X_{\underline{C}}) \leq Chr(G(T)) \leq \omega_1$ for any $\underline{C}$. Thus we will have to show that $Chr(X_{\underline{C}}) > \omega$ in the end.

By induction on $\delta \in S'$ (where S' denotes the accumulation points of S), we define the transitive ladder system $\underline{C}$ on $T_{<\delta}$ and hence the corresponding part of $X_{\underline{C}}$ on $T_{<\delta}$.

First, let $C_t = \emptyset$ for $t \in T_{<\min S'}$. Now, fix $\delta \in S'$ and suppose that we already defined a transitive ladder system $(C_t : t \in T_{<\delta})$ on $T_{<\delta}$. We extend this to $T_{<\delta^+}$ while preserving transitivity where δ^+ is the minimum of $S' \setminus (\delta + 1)$. Note that transitivity is necessarily preserved at limit steps. If $\delta \notin S$ then we let $C_t = \emptyset$ for $t \in T_{<\delta^+} \setminus T_{<\delta}$.

Suppose that $\delta \in S$ and let us define C_t for $t \in T_\delta$ first. Let $\{(A_\xi, f_\xi) : \xi < \mathfrak{c}\}$ denote a 1-1 enumeration of all the pairs (A, f) so that $A \in [T_{<\delta}]^\omega$, $f : A \to \omega$ and A satisfies

($\star$) for every $t \in A$ and $\varepsilon < \delta$ there are incomparable $s^0, s^1 \in A$ so that $s^i \geq t$ and $\max(s^i) > \varepsilon$ for $i < 2$.

By induction on $\xi < \mathfrak{c}$ we will find $t_\xi \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and sets $C_{t_\xi} \subseteq t_\xi^{\downarrow}$ so that

$$\left(C_t : t \in T_{<\delta} \cup \{t_\xi : \xi < \mathfrak{c}\}\right)$$

is still a transitive ladder system. We will let $C_t = \emptyset$ for $t \in T_\delta \setminus \{t_\xi : \xi < \mathfrak{c}\}$.

Fix a cofinal ω-type sequence $\{\delta_n : n \in \omega\}$ in δ. Suppose we defined $t_\zeta \in T_\delta$ and $C_{t_\zeta} \subseteq t_\zeta^{\downarrow}$ for $\zeta < \xi$. Define a map $\psi : 2^{<\omega} \to A_\xi$ and a partial map $\varphi : 2^{<\omega} \to A_\xi$ so that

(i) ψ and φ are order preserving injections and

$$\psi(x) \leq \varphi(x) \leq \psi(x^\frown i)$$

for $i < 2$ provided that $x \in \text{dom}(\varphi)$,

(ii) $\{\varphi(x \restriction k) : k \leq |x|, x \restriction k \in \mathrm{dom}(\varphi)\}$ is a complete graph in $T_{<\delta}$,

(iii) $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ are incomparable and contained in $A_\xi \setminus T_{<\delta_n}$,

(iv) <u>if</u> there is $t \in A_\xi$ such that $t \geq \psi(x)$, $f_\xi(t) = n$ and $\{\varphi(x \restriction k) : k < |x|, x \restriction k \in \mathrm{dom}(\varphi)\} \cup \{t\}$ is complete <u>then</u> $x \in \mathrm{dom}(\varphi)$ and $f_\xi(\varphi(x)) = n$ as well

for all $n \in \omega$ and $x \in 2^n$.

We define $\psi(x)$ and $\varphi(x)$ for $x \in 2^n$ by induction on $n \in \omega$. We set $\psi(\emptyset) \in A_\xi$ arbitrarily. If $\psi(x)$ is defined for some $x \in 2^n$ then check if

$$R_x = \{t \in A_\xi : t \geq \psi(x), f_\xi(t) = n \text{ and } \{\varphi(x \restriction k) : k < |x|, x \restriction k \in \mathrm{dom}(\varphi)\} \cup \{t\} \text{ is complete}\}$$

is empty or not. If $R_x \neq \emptyset$ then we put x into $\mathrm{dom}(\varphi)$ and pick $\varphi(x)$ from R_x arbitrarily; otherwise $x \notin \mathrm{dom}(\varphi)$. Now find incomparable $\psi(x^\frown 0), \psi(x^\frown 1) \in A_\xi$ above $\psi(x)$ so that $\max(\psi(x^\frown i)) \geq \delta_n$ and $\psi(x^\frown i) \geq \varphi(x)$ if $x \in \mathrm{dom}(\varphi)$; this can be done as A_ξ satisfies $(\star)$ above. This finishes the construction of ψ and φ.

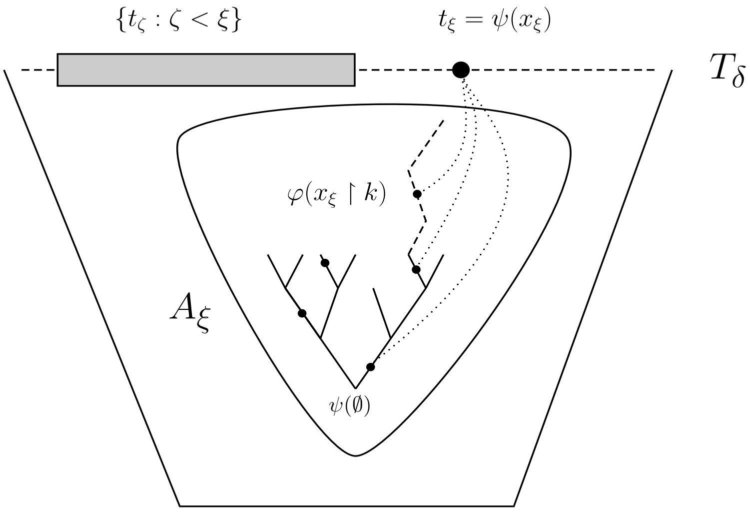


Figure 6.1: Step ξ in the induction.

We extend ψ to 2^ω in the obvious way:

$$\psi(x) = \bigcup\{\psi(x \restriction k) : k < \omega\} \cup \{\delta\}$$

for $x \in 2^\omega$; note that $\psi(x)$ is a closed subset of S by the second part of condition (iii) and hence $\psi(x) \in T_\delta$ for all $x \in 2^\omega$. Also, ψ remains 1-1 on 2^ω by the first part of condition (iii). Hence, we can find an $x_\xi \in 2^\omega$ such that $\psi(x_\xi) \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and we let $t_\xi = \psi(x_\xi)$. Finally, let

$$C_{t_\xi} = \big\{\varphi(x_\xi \restriction k) : k < \omega, x_\xi \restriction k \in \mathrm{dom}(\varphi)\big\}.$$

Transitivity of this extension is assured by condition (ii). This finishes the induction on $\xi < \mathfrak{c}$ and we have a transitive ladder system $\underline{C} = (C_t : t \in T_{\leq\delta})$ on $T_{\leq\delta}$. We now simply let $C_t = \emptyset$ for $t \in T_{<\delta^+} \setminus T_{\leq\delta}$.

This finishes step δ of the main induction and hence, in the end, we have a transitive ladder system $\underline{C}$ on T.

We are left to prove:

Claim 6.3.6. *$Chr(X_{\underline{C}}) > \omega$.*

Proof. Fix a colouring $f : T \to \omega$; we will find $s, t \in T$ so that $f(s) = f(t)$ and $s \in C_t$. Take a countable elementary submodel $M \prec H(\mathfrak{c}^+)$ (where $H(\mathfrak{c}^+)$ is the collection of sets with hereditary cardinality $\leq \mathfrak{c}$) so that $S, \underline{C}, f \in M$ and $\delta = M \cap \omega_1 \in S$; this can be done as S is stationary.

Consider the construction of $\{C_t : t \in T_\delta\}$. If we set $A = M \cap T$ then there must be a $\xi < \mathfrak{c}$ so that $(A, f \restriction A) = (A_\xi, f_\xi)$; M being an elementary submodel ensures that A satisfies property $(\star)$ as the branching of T reflects to M.

Our goal now is to prove that there is an $s \in C_{t_\xi}$ so that $f(s) = f(t_\xi)$; we let $n = f(t_\xi)$. Recall that in the definition of t_ξ we had two maps ψ and φ and t_ξ was of the form $\psi(x_\xi)$ for an $x_\xi \in 2^\omega$. C_{t_ξ} was defined to be $\{\varphi(x_\xi \restriction k) : k < \omega, x_\xi \restriction k \in \mathrm{dom}(\varphi)\}$.

Now, recall the definition of $\varphi(x_\xi \restriction n)$: we looked at the set

$$R_{x_\xi \restriction n} = \{s \in A_\xi : s \geq \psi(x_\xi \restriction n), f_\xi(s) = n \text{ and } \{\varphi(x_\xi \restriction k) : k < n, x_\xi \restriction k \in \mathrm{dom}(\varphi)\} \cup \{s\} \text{ is complete}\} \quad (6.1)$$

and if $R_{x_\xi \restriction n}$ was not empty the we chose $\varphi(x_\xi \restriction n) \in R_{x_\xi \restriction n}$; in particular, $f(s) = n$ for $s = \varphi(x_\xi \restriction n) \in C_{t_\xi}$ which would finish the proof.

Let us show that $R_{x_\xi \restriction n}$ is not empty. Let

$$R = \{s \in T : s \geq \psi(x_\xi \restriction n), f(s) = n \text{ and } \{\varphi(x_\xi \restriction k) : k < n, x_\xi \restriction k \in \mathrm{dom}(\varphi)\} \cup \{s\} \text{ is complete}\} \quad (6.2)$$

and note that R is in the model M and $R_{x_\xi \restriction n} = R \cap M$. Hence, by elementarity, it suffices to show that $R \neq \emptyset$. However, this is clear as $t_\xi \in R$.

□

This finishes the proof of the theorem. □

Let us remark that we cannot hope to find graphs of chromatic number $> \omega_1$ without uncountable infinitely connected subgraphs. Indeed, under $V = L$, the uncountable infinitely connected graph K_{ω,ω_1} embeds into every graph G with $Col(G) > \omega_1$ by a result of Komjáth [56, Theorem 3.1].

6.4 A highly disconnected variation

In Theorem 6.3.5, we constructed a graph X such that any uncountable set A *contained* two incomparable points s, t which are *separated in* A by a finite set F; in particular, there could be paths connecting s and t which avoid F by leaving A. The aim of this section is to refine the methods of Section 6.3 and produce a ladder system $\underline{C}$ on a tree T such that *any two $<_T$-incomparable vertices* are separated by a finite set in the graph $X_{\underline{C}}$ i.e. if $s, t \in T$ are incomparable then there is a finite set $F \subset T$ such that *any path P from s to t intersects F.* Note that this separation property is stronger than the lack of uncountable ω-connected sets.

We use the following notation: if T is a tree then

- let $supp(\underline{C}) = \{t \in T : |C_t| = \omega\}$ for any ladder system $\underline{C}$ on T,
- if $s \in T$ and $\varepsilon < ht(s)$ then $s \restriction \varepsilon$ denotes the unique element $r \in s^{\downarrow}$ with $ht(r) = \varepsilon$.

Let us introduce a somewhat technical property of ladder systems on trees. First, we need the following definition: we will say that a sequence $\underline{\eta} = (\eta_t : t \in T)$ is a *true ladder system* on a tree T iff $\eta_t = \{t\}$ for all successor $t \in T$ and η_t is a cofinal sequence of type ω in $t^{\downarrow}$ if t is limit.

Definition 6.4.1. *Suppose that $\underline{C}$ is a ladder system on a tree T. We say that $\underline{C}$ is* coherent *iff*

1. $C_s = C_t \cap s^{\downarrow}$

for all $t \in supp(\underline{C})$ and $s \in C_t$ with $|C_s| < \omega$ and there is a true ladder system $\underline{\eta}$ on T such that

2. $\eta_t \cap s^{\downarrow} \sqsubseteq \eta_s$,
3. $C_t \cap r^{\downarrow} = C_s \cap r^{\downarrow}$ *for* $r = s \restriction ht(\max_{<_T}(\eta_t \cap s^{\downarrow})) + 1$

for every $s, t \in supp(\underline{C})$ with $s \in C_t$.

The next lemma explains how transitivity and coherence of a ladder system $\underline{C}$ gives the desired separation property of the graph $X_{\underline{C}}$.

Lemma 6.4.2. *Suppose that T is a tree with no branching at limits. If $\underline{C}$ is a transitive and coherent ladder system on T then any two $<_T$-incomparable points are separated by a finite set in $X_{\underline{C}}$.*

We use the following notation in the proof: if $s, t \in T$ then $\Delta(s, t)$ denotes the maximal common initial segment of s and t.

Proof. Fix a point $t' \in T$ and we prove that for any $t \in T$ which is incomparable with t' there is a finite set F_t which separates t and t'.

We will actually prove that the following choice of F_t works: let $F_t = C_t$ if $t \notin supp(C)$ and let

$$F_t = C_t \cap r_t^{\downarrow} \text{ where } r_t = t \restriction ht(\min_{<_T}\{r \in \eta_t : r > \Delta(t, t')\}) + 1$$

if $t \in supp(\underline{C})$. The proof is done by induction on $ht(t)$.

It is clear that if $t \notin supp(\underline{C})$ then F_t separates t and t' by transitivity and Lemma 6.3.3.

Suppose that $t \in supp(\underline{C})$; we note that $r_t < t$ as η_t is nontrivial and $\Delta(t, t') < t$ and hence F_t is finite. Now, suppose that $P = \{p_0 \dots p_n\}$ is a path from t to t' which avoids F_t; we can suppose that P has minimal length and is the union of two monotone paths by Lemma 6.3.3. We have $p_0 = t > p_1 \in C_t$ and $p_1 > \Delta(t, t')$ and hence $\Delta(t, t') = \Delta(p_1, t')$. We let $s = p_1$ and note that $p_2 \in C_s \setminus C_t$ as P has minimal length and $\underline{C}$ is transitive. If $|C_s| < \omega$ then by assumption $C_t \cap s^{\downarrow} = C_s$ which contradicts $p_2 \in C_s \setminus C_t$.

We conclude that $s \in supp(\underline{C})$. As F_s separates s and t' (by the inductive hypothesis) it suffices to prove that $F_s = F_t$. By definition

$$r_s = s \restriction ht(\min_{<_T}\{r \in \eta_s : r > \Delta(s, t')\}) + 1.$$

As $t, s \in supp(\underline{C})$ we have $\eta_t \cap s^{\downarrow} \sqsubseteq \eta_s$ by coherence. Hence, by $r_t < s$ and $\Delta(t, t') = \Delta(s, t')$ we must have $r_s = r_t$. Finally, by coherence again, we have $F_t = C_t \cap r_t{}^{\downarrow} = C_s \cap r_s{}^{\downarrow} = F_s$. □

We are ready to prove the main result of this section.

Theorem 6.4.3. *Fix a stationary, costationary $S \subset \omega_1$ and let $T = T(S)$. Then there is a subgraph X of $G(T)$ such that $Chr(X) = \omega_1$ and any two $<_T$-incomparable points are separated by a finite set in X. In particular, every uncountable set $A \subseteq T$ contains two vertices which are separated by a finite set in X.*

Proof. It suffices to find a transitive and coherent ladder system $\underline{C}$ on T such that $Chr(X_{\underline{C}}) > \omega$; indeed, T does not branch at limits (and every uncountable $A \subseteq T$ contains two incomparable points) hence Lemma 6.4.2 can be applied.

We define a true ladder system $\underline{\eta}$ on T first: fix a true ladder system $\{\nu_\delta : \delta \in \omega_1\}$ on ω_1 and let

$$\eta_t = \{t \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta\} \text{ where } \delta = \max(t)$$

for any limit $t \in T$ and let $\eta_t = \{t\}$ for any successor $t \in T$.

Let $D = (S \cap S')'$ and let δ^+ denote $\min D \setminus (\delta + 1)$ for $\delta \in D$. By induction on $\delta \in D$, we define a transitive ladder system $\underline{C}$ on $T_{<\delta}$, and hence the corresponding graph on $T_{<\delta}$, so that $\underline{C}$ is coherent and its coherence is witnessed by $\underline{\eta}$.

First, let $C_t = \emptyset$ for $t \in T_{<\min D}$. Now, suppose we defined $(C_t)_{t \in T_{<\delta}}$ for some $\delta \in D$. We define $\underline{C}$ on $T_{<\delta^+}$ in two steps: first we define C_t for $t \in T_\delta$ so that $(C_t)_{t \in T_{\leq \delta}}$ is still transitive and coherent and then extend to $T_{<\delta^+}$ in the trivial way: we let $C_t = \emptyset$ for $t \in T_{<\delta^+} \setminus T_{\leq \delta}$.

We can suppose $\delta \in S$ otherwise $T_\delta = \emptyset$. Let $\{((A^\xi_n)_{n\in\omega}, f_\xi) : \xi < \mathfrak{c}\}$ denote a 1-1 enumeration of all pairs $((A_n)_{n\in\omega}, f)$ where

1. $A_n \subset A_{n+1} \in [T_{<\delta}]^\omega$ for $n < \omega$,
2. for every $t \in A_n$ and $\varepsilon < \delta_n = \sup\{\max(s) : s \in A_n\}$ there are incomparable $s^0, s^1 \in A_n$ so that $s^i \geq t$ and $\max(s^i) > \varepsilon$ for $i < 2$,
3. $f : A \to \omega$ where $A = \bigcup\{A_n : n \in \omega\}$,
4. $(\delta_n)_{n\in\omega}$ is a strictly increasing cofinal sequence in δ.

By induction on $\xi < \mathfrak{c}$, we define $t_\xi \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and $C_{t_\xi} \subseteq t^{\downarrow}_\xi$ while preserving transitivity and coherence. Suppose we defined $t_\zeta \in T_\delta$ and $C_{t_\zeta} \subseteq t^{\downarrow}_\zeta$ for $\zeta < \xi$.

We let $A_\xi = \bigcup\{A^\xi_n : n \in \omega\}$ and $\delta^\xi_n = \sup\{\max(s) : t \in A^\xi_n\}$. Also, let

$$\varepsilon_n = \max(\{\delta^\xi_n\} \cup (\nu_\delta \cap \delta^\xi_{n+1}))$$

for $n \in \omega$ and $\varepsilon_{-1} = \max(\nu_\delta \cap \delta^\xi_0)$. Finally, let $\{l_n : n \in \omega\}$ enumerate each natural number infinitely many times.

Define a map $\psi : 2^{<\omega} \to A_\xi$ and a partial map $\varphi : 2^{<\omega} \to A_\xi$ so that

(i) ψ and φ are order preserving injections and

$$\psi(x) \leq \varphi(x) \leq \psi(x^\frown i)$$

for $i < 2$ provided that $x \in \mathrm{dom}(\varphi)$,

(ii) $\{\varphi(x \upharpoonright k) : k \leq |x|, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$ is a complete graph in $T_{<\delta}$,

(iii) $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ are incomparable and contained in $A^\xi_{n+1} \setminus T_{<\varepsilon_n}$,

(iv) if there is an $s \in A^\xi_n$ such that

(a) $s \geq \psi(x)$, $f_\xi(s) = l_n$,

(b) $\{\varphi(x \upharpoonright k) : k < |x|, x \upharpoonright k \in \mathrm{dom}(\varphi)\} \cup \{s\}$ is complete,

(c) $\nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)}$ and

$$C_s \cap r^\downarrow = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$$

for $r = s \cap (\varepsilon_{n-1} + 1)$ if C_s is infinite,

(d) $C_s = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$ if C_s is finite

then $x \in \mathrm{dom}(\varphi)$ and $s = \varphi(x)$ satisfies (a)-(d) as well,

(v) if (iv) fails then $x \notin \mathrm{dom}(\varphi)$

for all $n \in \omega$ and $x \in 2^n$.

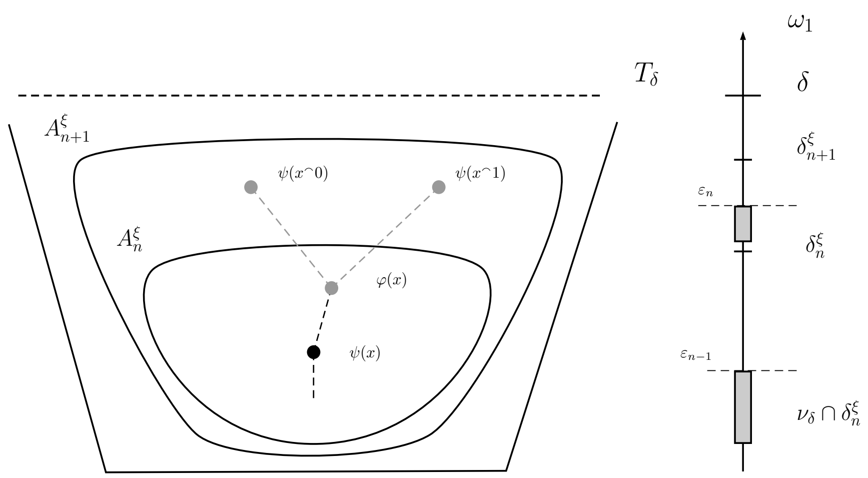


Figure 6.2: Extending the maps φ and ψ.

We define $\psi(x)$ and $\varphi(x)$ for $x \in 2^n$ by induction on $n \in \omega$. We let $\psi(\emptyset) \in A^\xi_0 \setminus T_{\leq \varepsilon_{-1}}$ arbitrarily. Given $\psi(x)$, we consider the set R^ξ_x of all elements $s \in A^\xi_n$ such that

(a') $s \geq \psi(x)$, $f_\xi(s) = l_n$,

(b') $\{\varphi(x \upharpoonright k) : k < |x|, x \upharpoonright k \in \mathrm{dom}(\varphi)\} \cup \{s\}$ is complete,

(c') $\nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)}$ and

$$C_s \cap r^\downarrow = \{\varphi(x \upharpoonright k) : k < n, x \upharpoonright k \in \mathrm{dom}(\varphi)\}$$

for $r = s \cap (\varepsilon_{n-1} + 1)$ if C_s is infinite, (d') $C_s = \{\varphi(x \restriction k) : k < n, x \restriction k \in \mathrm{dom}(\varphi)\}$ if C_s is finite.

If R^ξ_x is not empty then we set $x \in \mathrm{dom}(\varphi)$ and choose an arbitrary $\varphi(x) \in R^\xi_x$. Otherwise, $x \notin \mathrm{dom}(\varphi)$. Now, we simply pick $\psi(x^\frown i)$ for $i < 2$ satisfying conditions (i) and (iii) by applying condition 2. for A^ξ_{n+1}. This finishes the construction of ψ and φ.

We extend ψ to 2^ω in the obvious way:

$$\psi(x) = \bigcup\{\psi(x \restriction k) : k < \omega\} \cup \{\delta\}$$

for $x \in 2^\omega$; note that $\psi(x)$ is a closed subset of S by the second part of condition (iii) hence $\psi(x) \in T_\delta$ for all $x \in 2^\omega$. Also, ψ remains 1-1 on 2^ω by the first part of condition (iii). Hence, we can find an $x_\xi \in 2^\omega$ such that $\psi(x_\xi) \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and we let $t_\xi = \psi(x_\xi)$. Finally, let

$$C_{t_\xi} = \big\{\varphi(x_\xi \restriction k) : k < \omega, x_\xi \restriction k \in \mathrm{dom}(\varphi)\big\}.$$

This finishes the induction on $\xi < \mathfrak{c}$ and we define $C_t = \emptyset$ for $t \in T_\delta \setminus \{t_\xi : \xi < \mathfrak{c}\}$.

Claim 6.4.4. *$\{C_t : t \in T_{\leq\delta}\}$ is transitive and coherent.*

Proof. Transitivity is assured by condition (ii). We check that $\underline{\eta}$ witnesses that $\{C_t : t \in T_{\leq\delta}\}$ is coherent. Fix $\xi < \mathfrak{c}$ and $n < \omega$ such that $x_\xi \restriction n \in \mathrm{dom}(\varphi)$ i.e. $s = \varphi(x_\xi \restriction n) \in C_{t_\xi}$.

If C_s is finite then we need that $C_s = C_{t_\xi} \cap s^\downarrow$; this is assured by condition (iv)-(d) above. Suppose that C_s is infinite; we need to check that

$$\eta_{t_\xi} \cap s^\downarrow \sqsubseteq \eta_s \text{ and } C_{t_\xi} \cap r^\downarrow = C_s \cap r^\downarrow$$

for $r = s \restriction ht(\max\limits_{<_T}(\eta_{t_\xi} \cap s^\downarrow)) + 1$. Recall that

$$\eta_{t_\xi} = \{t_\xi \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta\} \text{ and } \eta_s = \{s \cap (\varepsilon + 1) : \varepsilon \in \nu_{\max(s)}\}.$$

Note that $u \in \eta_{t_\xi} \cap s^\downarrow$ iff $u = t_\xi \cap (\varepsilon + 1)$ for some $\varepsilon \in \nu_\delta \cap \max(s)$. Furthermore, $\nu_\delta \cap \max(s) = \nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)}$ by the choice of s and condition (iv)-(c). Hence, as $s \sqsubseteq t_\xi$, we get that $\eta_{t_\xi} \cap s^\downarrow \sqsubseteq \eta_s$.

Finally, note that condition (iv)-(c) says that C_s and C_{t_ξ} agree below $s \cap (\varepsilon_{n-1}+1)$ and $\max(\eta_{t_\xi} \cap s^\downarrow) \leq (s \cap \varepsilon_{n-1} + 1)$. This shows $C_{t_\xi} \cap r^\downarrow = C_s \cap r^\downarrow$. □

This finishes our main induction and, as transitivity and coherence are preserved at limit steps, we constructed a transitive and coherent ladder system $\underline{C}$ on the tree T. It is left to prove

Claim 6.4.5. *$Chr(X_{\underline{C}}) > \omega$.*

Proof. Fix a colouring $f : T \to \omega$; we will find $s, t \in T$ such that $f(s) = f(t)$ and $s \in C_t$. We fix an increasing sequence $(M_n : n \in \omega)$ of countable elementary submodels of $H(\mathfrak{c}^+)$ so that $S, \underline{C}, f \in M_n$ for all $n \in \omega$ and $\delta = M \cap \omega_1 \in S$ for $M = \bigcup\{M_n : n \in \omega\}$.

We consider the construction of $\{C_t : t \in T_\delta\}$. There is a $\xi < \mathfrak{c}$ so that $A^\xi_n = T \cap M_n$ for all $n \in \omega$ and $f_\xi = f \restriction (M \cap T)$. Our goal is to show that there is an $s \in C_{t_\xi}$ such that $f(s) = f(t_\xi)$; we let $l = f(t_\xi)$. Recall that there is an $x = x_\xi \in 2^\omega$ so that $t_\xi = \psi(x)$ and $C_{t_\xi} = \{\varphi(x \restriction k) : k < \omega, x \restriction k \in \mathrm{dom}(\varphi)\}$; we will show that there is an $n < \omega$ such that $x \restriction n \in \mathrm{dom}(\varphi)$ and $f(\varphi(x \restriction n)) = l$, which finishes the proof.

We first show that C_{t_ξ} is infinite, equivalently that there are infinitely many $n \in \omega$ such that $x \restriction n \in \mathrm{dom}(\varphi)$. Suppose otherwise i.e. there is an $n_0 < \omega$ such that $x \restriction n \notin \mathrm{dom}(\varphi)$ for all $n \in \omega \setminus n_0$. Let n be the minimal element of $\omega \setminus n_0$ such that $l = l_n$.

Now, recall how we tried to construct $\varphi(x \restriction n)$: we looked at the set $R^\xi_{x\restriction n}$ (elements from A^ξ_n which satisfied conditions (a')-(d')) and if $R^\xi_{x\restriction n}$ was not empty then we put $x \restriction n \in \mathrm{dom}(\varphi)$ and chose $\varphi(x \restriction n) \in R^\xi_{x\restriction n}$. Let us show that $R^\xi_{x\restriction n}$ is not empty. Let

$$R = \{s \in T : s \geq \psi(x \restriction n), f(s) = l_n \text{ and } C_s = \{\varphi(x \restriction k) : k < n, x \restriction k \in \mathrm{dom}(\varphi)\}\}$$

and note that R is in the model M_n and $R \cap M_n \subseteq R^\xi_{x\restriction n}$. Hence, by elementarity, it suffices to show that $R \neq \emptyset$. However, this is clear as $t_\xi \in R$. This contradicts $x \restriction n \notin \mathrm{dom}(\varphi)$ and in turn shows that C_{t_ξ} is infinite.

Now, with a quite similar argument, we will show that $x \restriction n \in \mathrm{dom}(\varphi)$ for the minimal $n \in \omega$ such that $l = l_n$; this finishes the proof of the claim as $s = \varphi(x \restriction n) \in C_{t_\xi}$ and $f(s) = l = f(t_\xi)$. Again, we look at $R^\xi_{x\restriction n}$ and prove that $R^\xi_{x\restriction n} \neq \emptyset$. Consider

$$\begin{aligned} R = \{s \in T \cap \mathit{supp}(\underline{C}) : s \geq \psi(x \restriction n), f(s) = l_n,& \\ \{\varphi(x \restriction k) : k < n, x \restriction k \in \mathrm{dom}(\varphi)\} \cup \{s\} \text{ is complete},& \\ \nu_\delta \cap \varepsilon_{n-1} \sqsubseteq \nu_{\max(s)} \text{ and } C_s \cap r^\downarrow = \{\varphi(x \restriction k) : k < n, x \restriction k \in \mathrm{dom}(\varphi)\} \text{ for } r = s \cap (\varepsilon_{n-1}+1)\}.& \end{aligned}$$

It is clear that $R \in M_n$ and $R \cap M_n \subseteq R^\xi_{x\restriction n}$ hence it suffices to show, by elementarity, that $R \neq \emptyset$. However, $t_\xi \in R$.

□

This finishes the proof of the theorem. □

6.5 A new triangle-free graph in ZFC

In this section, we adapt ideas of A. Hajnal and P. Komjáth [42] to our setting and construct a ladder system $\underline{C}$ on $T = T(S)$ (with $S \subset \omega_1$ stationary) so that $X_{\underline{C}}$ is uncountably chromatic, triangle free and contains no copies of the graph $H_{\omega,\omega+2}$. A graph with these particular properties is constructed in [42] but using the Continuum Hypothesis and later in ZFC in [64, Theorem 10]. Our construction is also purely in ZFC.

Definition 6.5.1. *Suppose that T is a tree. A cycle $x_0, x_1, ...x_n = x_0$ in $G(T)$ is* special *if it is the union of two $<_T$-monotone paths.*

Note that every triangle is a special cycle. Our aim is to construct a graph of the form $X_{\underline{C}}$ without special cycles.

Definition 6.5.2. *Suppose that T is a tree of the form $T(S)$, X is a subgraph of $G(T)$ and $\gamma < \omega_1$. We say that a vertex $v \in T$ is γ-*covered *in X iff there exists a point $w \in T_{\leq\gamma}$ and a monotone path from w to v in X.*

A ladder system $\underline{C}$ on T is sparse *iff s is not $\max(r)$-covered in $X_{\underline{C}}$ for each $t \in T$ and $r, s \in C_t$ with $r < s$.*

Note that if $\underline{C}$ is sparse then C_t is independent in $X_{\underline{C}}$ for all $t \in T$ and hence $X_{\underline{C}}$ is triangle free. The following was essentially proved in [42] and motivates the definitions above:

Lemma 6.5.3. *Suppose that $\underline{C}$ is a ladder system on T. Then*

1. *if $\underline{C}$ is sparse then $X_{\underline{C}}$ contains no special cycles,*
2. *if $X_{\underline{C}}$ contains no special cycles then $X_{\underline{C}}$ contains no triangles or copies of $H_{\omega,\omega+2}$.*

Proof. (1) Suppose that $x_0, x_1, \dots x_n = x_0$ is a special cycle with $\max(x_i) = \alpha_i$. Hence, there is $i < n$ so that $\alpha_j < \alpha_i$ if $i \neq j \leq n$. In particular, $x_{i-1}, x_{i+1} \in C_{x_i}$ and without loss of generality $\alpha_{i-1} < \alpha_{i+1} < \alpha_i$. However, this implies that x_{i+1} is α_{i-1}-covered, witnessed by the path $x_n, x_{n-1}, \dots x_{i+1}$, which contradicts that $\underline{C}$ is sparse.

(2) It is clear that every triangle is a special cycle. Now, suppose that $\{x_i, y_i, z, z' : i < \omega\}$ is a subgraph of $X_{\underline{C}}$ isomorphic to $H_{\omega,\omega+2}$ i.e. the following pairs of points are edges

$$\big\{\{x_i, y_j\}, \{x_i, z\}, \{x_i, z'\} : i \leq j \in \mathbb{N}\big\}.$$

First, as x_i and x_j have infinitely many common neighbours (for $i < j < \omega$) they must be $<_T$-comparable; hence we can suppose that $x_0 <_T x_1 <_T ....$ Second, either z or z' is $<_T$-below infinitely many x_i so we might as well suppose that $z <_T x_i$ for all $i < \omega$. Finally, we have $z <_T x_0 <_T x_1$ and x_0, x_1 have infinitely many common neighbours of the form y_j with $\max(y_j) > \alpha_1$. In particular, we can find special cycles of length 4 which contradicts our assumption. $\square$

Hence, we will aim at constructing sparse ladder systems $\underline{C}$ such that the corresponding graphs $X_{\underline{C}}$ are uncountably chromatic. Before that, we need the following

Lemma 6.5.4. *Fix a stationary $S \subset \omega_1$ and let $T = T(S)$. Suppose that X is a subgraph of $G(T)$ and $f : T \to \omega$. Then there is $\delta \in \omega_1$ and $t \in T_\delta$ so that for every $n \in \omega$ either:*

1. *for every $r \geq t$ and every $\gamma \in \omega_1$ there is an $s \geq r$ with $f(s) = n$ which is* not *γ-covered in X, or*
2. *every $r \geq t$ with $f(r) = n$ is δ-covered in X.*

We will say that the vertex t *decides* f.

Proof. Take a countable elementary submodel $M \prec H(\mathfrak{c}^+)$ with $f, X, S \in M$ so that $M \cap \omega_1 = \delta \in S$. Fix a cofinal sequence $\{\delta_n : n \in \omega\}$ of type ω in δ.

Now, construct a sequence $t_0 \leq .... \leq t_n \leq ...$ in $M \cap T$ so that $\max(t_n) \geq \delta_n$ and for every $n \in \omega$ either

(i) for every $r \geq t_{n+1}$ and every $\gamma \in \omega_1$ there is an $s \geq r$ with $f(s) = n$ which is *not* γ-covered in X, or

(ii) there is a $\gamma_n \in \delta$ so that every $r \geq t_{n+1}$ with $f(r) = n$ is γ_n-covered in X.

We can pick $t_0 \in M \cap T$ with $\max(t_0) \geq \delta_0$ arbitrarily. Given $t_n \in M$ we select $t'_{n+1} \in M$ above t_n so that $\max(t'_{n+1}) \geq \delta_{n+1}$. If the choice $t_{n+1} = t'_{n+1}$ satisfies (i) from above then we are done; otherwise, there is $t_{n+1} \geq t'_{n+1}$ and $\gamma = \gamma_n$ so that every $r \geq t_{n+1}$ with $f(r) = n$ is γ_n-covered. t_{n+1} and γ_n can be chosen in M by elementarity so we are done.

Now, let $t = \bigcup\{t_n : n \in \omega\} \cup \{\delta\}$ and note that $t \in T_\delta$; we claim that this t decides f. Indeed, as $t \geq t_n$ for all $n \in \omega$ and t_n satisfies (i) or (ii), t must satisfy either 1. or 2. respectively.

$\square$

We are ready to prove the main result of this section:

Theorem 6.5.5. *Fix a stationary $S \subseteq \omega_1$ and let $T = T(S)$. Then there is subgraph X of $G(T)$ with $Chr(X) = \omega_1$ such that X contains no special cycles; in particular, X contains no triangles or copies of $H_{\omega,\omega+2}$.*

Proof. It suffices to construct a sparse ladder system $\underline{C}$ on T so that $Chr(X_{\underline{C}}) = \omega_1$; indeed, by Lemma 6.5.3, a sparse ladder system $\underline{C}$ induces a graph $X_{\underline{C}}$ on T with no special cycles and hence no triangles or copies of $H_{\omega,\omega+2}$.

We define a sparse ladder system $(C_t : t \in T_{<\delta})$ by induction on $\delta \in S'$ and so that $C_t = \emptyset$ if $t \in T$ is a successor. First, let $C_t = \emptyset$ for $t \in T_{<\min S'}$. Suppose we constructed C_t for $t \in T_{<\delta}$ and we now extend this ladder system to $T_{<\delta^+}$ where δ^+ is the minimum of $S' \setminus (\delta + 1)$ in two steps. First, we define C_t for $t \in T_\delta$ and then let $C_t = \emptyset$ for $t \in T_{<\delta^+} \setminus T_{\leq\delta}$. We may suppose $\delta \in S$, otherwise $T_\delta = \emptyset$.

Let $\{(A_\xi, f_\xi, t_{0\xi}) : \xi < \mathfrak{c}\}$ denote a 1-1 enumeration of all triples (A, f, t_0) with $A \in [T_{<\delta}]^\omega$, $f : A \to \omega$ and $t_0 \in A$ so that

($\star$) for every $t \in A$ and $\varepsilon < \delta$ there are incomparable $s^0, s^1 \in A$ so that $s^i \geq t$ and $\max(s^i) \geq \varepsilon$ for $i < 2$.

By induction on $\xi < \mathfrak{c}$ we define $t_\xi \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and $C_{t_\xi} \subseteq t_\xi^\downarrow$ (while preserving that the ladder system is sparse). Suppose we have $\{t_\zeta : \zeta < \xi\}$ defined and consider the triple $(A_\xi, f_\xi, t_{0\xi})$. Fix a cofinal increasing sequence $\{\delta_n : n \in \omega\}$ of type ω in δ.

We define a map $\psi : 2^{<\omega} \to A_\xi$ and a partial map $\varphi : 2^{<\omega} \to A_\xi$ so that

(i) ψ and φ are order preserving injections and

$$t_{0\xi} < \psi(x) \leq \varphi(x) \leq \psi(x^\frown i)$$

for $i < 2$ provided that $x \in \mathrm{dom}(\varphi)$,

(ii) $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ are incomparable and contained in $A_\xi \setminus T_{<\delta_n}$,

(iii) $\varphi(x)$ is not $\max(\psi(x))$-covered,

(iv) <u>if</u> there is an $s \in A_\xi$ such that

  (a) $s \geq \psi(x), f_\xi(s) = n$,

  (b) s is not $\max(\psi(x))$-covered

<u>then</u> $x \in \mathrm{dom}(\varphi)$ and $\varphi(x)$ satisfies (a)-(b) as well

for all $x \in 2^n$ and $n \in \omega$.

We define $\psi(x)$ and $\varphi(x)$ for $x \in 2^n$ by induction on $n \in \omega$. We select $\psi(\emptyset) > t_{0\xi}$ arbitrarily in A_ξ. Given $\psi(x)$ for $x \in 2^n$ we look at the set

$$R_x^\xi = \{s \in A_\xi : s \geq \psi(x), f_\xi(s) = n \text{ and } s \text{ is not } \max(\psi(x))\text{-covered}\}.$$

If R^ξ_x is not empty then let $x \in \mathrm{dom}(\varphi)$ and pick any $\varphi(x) \in R^\xi_x$; otherwise $x \notin \mathrm{dom}(\varphi)$. Now, using condition ($\star$) of A_ξ, select incomparable $\psi(x^\frown 0)$ and $\psi(x^\frown 1)$ so that conditions (i)-(ii) are satisfied. This finishes the construction of ψ and φ.

Now extend ψ to 2^ω in the obvious way:

$$\psi(x) = \bigcup\{\psi(x \restriction k) : k < \omega\} \cup \{\delta\}$$

for $x \in 2^\omega$; note that $\psi(x)$ is a closed subset of S by the second part of condition (ii) and hence $\psi(x) \in T_\delta$ for all $x \in 2^\omega$. Also, ψ remains 1-1 on 2^ω by the first part of condition (ii). Hence, we can find an $x_\xi \in 2^\omega$ such that $\psi(x_\xi) \in T_\delta \setminus \{t_\zeta : \zeta < \xi\}$ and we let $t_\xi = \psi(x_\xi)$. Finally, let

$$C_{t_\xi} = \big\{\varphi(x_\xi \restriction k) : k < \omega, x_\xi \restriction k \in \mathrm{dom}(\varphi)\big\}.$$

Note that condition (iii) ensures that C_{t_ξ} is sparse. This finishes the induction on $\xi < \mathfrak{c}$ and in turn the induction on $\delta \in S'$.

We are left to prove

Claim 6.5.6. *$Chr(X_{\underline{C}}) > \omega$.*

Proof. Fix a colouring $f : T \to \omega$; we will find $s, t \in T$ so that $f(s) = f(t)$ and $s \in C_t$. Take a countable elementary submodel $M \prec H(\mathfrak{c}^+)$ so that $S, \underline{C}, f \in M$ and $\delta = M \cap \omega_1 \in S$. By Lemma 6.5.4, we can find $t_0 \in M \cap T$ so that t_0 decides f.

Now, consider the construction of $\{C_t : t \in T_\delta\}$; note that there is a $\xi < \mathfrak{c}$ so that $(A_\xi, f_\xi, t_{0\xi}) = (A, f \restriction A, t_0)$ where $A = T \cap M$. We will show that there is $s \in C_{t_\xi}$ with $f(s) = f(t_\xi)$.

Let $f(t_\xi) = n$.

Observation 6.5.7. *For every $r \geq t_0$ and every $\gamma \in \omega_1$ there is an $s \geq r$ with $f(s) = n$ which is* not *γ-covered .*

Proof. Recall that $t_0 = t_{0\xi}$ decides f, so if the above statement fails then every $r \geq t_0$ with $f(r) = n$ is $\max(t_0)$-covered. In particular, t_ξ is $\max(t_0)$-covered. However, this implies that C_{t_ξ} is not empty and there is $s \in C_{t_\xi}$ which is $\max(t_0)$-covered (note that $s > t_0$ for all $s \in C_{t_\xi}$). However, every $s \in C_{t_\xi}$ is not $\max(t_{0\xi})$-covered by conditions (i) and (iii) above; this contradiction finishes the proof. □

Recall that there is an $x \in 2^\omega$ such that $t_\xi = \psi(x)$ and $C_{t_\xi} = \{\varphi(x \restriction k) : k < \omega, x \restriction k \in \mathrm{dom}(\varphi)\}$. Our aim is to show that $x \restriction n \in \mathrm{dom}(\varphi)$ and hence $f(s) = f(t_\xi)$ for $s = \varphi(x \restriction n) \in C_{t_\xi}$. Thus we need to prove that

$$R^\xi_{x\restriction n} = \{s \in A_\xi : s \geq \psi(x \restriction n), f_\xi(s) = n \text{ and } s \text{ is not } \max(\psi(x \restriction n))\text{-covered}\}$$

is not empty.

Let

$$R = \{s \in T : s \geq \psi(x \restriction n), f(s) = n \text{ and } s \text{ is not } \max(\psi(x \restriction n))\text{-covered}\}$$

and note that $R^\xi_{x\restriction n} = R \cap M$ and $R \in M$. Hence, by elementarity, it suffices to show that $R \neq \emptyset$. This clearly follows from Observation 6.5.7 applied to $r = \psi(x \restriction n)$ and $\gamma = \max(\psi(x \restriction n))$.

□

This finishes the proof of the theorem. □

Let us remark that sparse and transitive ladder systems represent two extremes in the spectrum of subgraphs of $G(T)$; if $\underline{C}$ is sparse then C_t is an independent set while if $\underline{C}$ is transitive then C_t is a complete subgraph.

6.6 More on trees and ladders

We would like to point out that some of the graphs defined in our paper satisfy strong *partition properties*. If T is a tree and X is a subgraph of $G(T)$ then we write

$$X \to (K_{\omega+1})^1_\omega$$

iff for every colouring $f : T \to \omega$ there is an $n \in \omega$ and a set $A \subseteq T \cap f^{-1}(n)$ of $<_T$-order type $\omega + 1$ such that A spans a complete graph (i.e. A is a monochromatic copy of $K_{\omega+1}$). Clearly, $X \to (K_{\omega+1})^1_\omega$ implies that $Chr(X) > \omega$ but not necessarily the other way; indeed, as seen in Theorem 6.5.5, there are even triangle free subgraphs of $G(T)$ (for some T) which are uncountably chromatic.

Let us first show that satisfying the above partition property or having large chromatic number are equivalent for *transitive* ladder systems.

Proposition 6.6.1. *Suppose that T is a tree and $\underline{C}$ is a ladder system on T. If $\underline{C}$ is transitive and $Chr(X_{\underline{C}}) > \omega$ then*

$$X_{\underline{C}} \to (K_{\omega+1})^1_\omega.$$

Proof. Fix an $f : T \to \omega$; we will show that there is an $n \in \omega$ and $t \in f^{-1}(n)$ so that $A = C_t \cap f^{-1}(n)$ is infinite hence, by transitivity, $A \cup \{t\}$ gives a monochromatic copy of $K_{\omega+1}$ in $X_{\underline{C}}$.

Suppose otherwise i.e. $C_t \cap f^{-1}(n)$ is finite for every $t \in T$ with $f(t) = n$. We can define a new colouring $g : T \to \omega \times \omega$ using induction on the height so that $g(t) = (f(t), g_1(t))$ where $g_1(t) = \max\{g_1(s) : s \in C_t \cap f^{-1}(n)\} + 1$ with $n = f(t)$. It is easy to see that g witnesses $Chr(X_{\underline{C}}) \leq \omega$ which is a contradiction. □

The above proposition is nicely complemented by

Observation 6.6.2. *If T is a tree of height ω_1 and $\underline{C}$ is a ladder system on T then there is no complete subgraph in $X_{\underline{C}}$ of $<_T$-order type $\omega + 2$.*

Recall that in the proof of Theorem 6.4.3, we used a ladder system on ω_1 (denoted by $\underline{\nu}$ there) to define another ladder system (denoted by $\underline{\eta}$) on $T = T(S)$ for $S \subseteq \omega_1$ in a very natural way. We did not consider the subgraph of $G(T)$ corresponding to $\underline{\eta}$ at that point so let us present a result here.

Proposition 6.6.3. *Suppose that $S \subseteq \omega_1$ is stationary and let $T = T(S)$. Fix a true ladder system $\underline{\nu} = \{\nu_\delta : \delta \in Lim(\omega_1)\}$ on ω_1. Let*

$$C_t = \{t \cap (\varepsilon + 1) : \varepsilon \in \nu_\delta\}$$

for any limit $t \in T_\delta$ and $\delta \in S$ and let $C_t = \emptyset$ otherwise. Then

$$X_{\underline{C}} \to (K_{\omega+1})^1_\omega.$$

Proof. Let $f : T \to \omega$. We say that $D \subseteq T$ is *dense above* $t \in T$ iff for every $s \geq t$ there is $r \in D$ such that $r \geq s$. D is *empty above* t iff $s \notin D$ for every $s \geq t$.

Claim 6.6.4. *There is t_0 in T so that $f^{-1}(k)$ is either empty or dense above t_0 for every $k \in \omega$.*

Proof. The proof is very similar to the argument seen in Lemma 6.5.4 so we will be brief here. Take a countable elementary submodel $M \prec H(\mathfrak{c}^+)$ with $f, S \in M$ so that $\delta = M \cap \omega_1 \in S$. Build a sequence $s_0 \leq s_1 \leq \dots$ in $T \cap M$ so that $(\max(s_k) : k \in \omega)$ is a cofinal ω-type sequence in δ and $f^{-1}(k)$ is either empty or dense above s_k. It is easy to see that

$$t_0 = \bigcup\{s_k : k \in \omega\} \cup \{\delta\}$$

is in T and satisfies the claim. $\square$

Take a countable elementary submodel $M \prec H(\mathfrak{c}^+)$ with $t_0, f, S, \underline{\nu} \in M$ and $\delta = M \cap \omega_1 \in S$. Let $\{k_n : n \in \omega\}$ enumerate those $k \in \omega$ so that $f^{-1}(k)$ is dense above t_0 (or equivalently, not empty above t_0), each ω times. Let $(\delta_n : n \in \omega)$ be an arbitrary cofinal ω-type sequence in δ.

We construct $t_0 \leq t_1 \leq .... \leq t_n \leq ...$ in $T \cap M$ so that

1. $\max(t_{n+1}) \geq \delta_n$,
2. $t_{n+1} \cap (\varepsilon_n + 1) = t_n$ for $\varepsilon_n = \min \nu_\delta \setminus (\max(t_n) + 1)$,
3. <u>if</u> there is $t \geq t_n$ in $T \cap M$ such that
   (a) $\max(t) \geq \delta_n$, $t \cap (\varepsilon_n + 1) = t_n$,
   (b) $\{t_i : i \leq n\} \subseteq C_t$ and
   (c) $f(t) = k_n$

   <u>then</u> $t = t_{n+1}$ satisfies (a)-(c) as well.

Let $t = \bigcup\{t_n : n \in \omega\} \cup \{\delta\}$; note that $t \in T_\delta$ and $t \geq t_n$ for all $n \in \omega$. Also, (2) ensures that $t_n \in C_t$ for $n \in \omega$ as $t_n = t \cap (\varepsilon_n + 1)$ and $\varepsilon_n \in \nu_\delta$.

Let $k = f(t)$; as $t \geq t_0$ we know that $f^{-1}(k)$ is dense above t_0. We claim that

$$A = \{t_{n+1} : k = k_n, n \in \omega\} \cup \{t\}$$

is a complete subgraph $X_{\underline{C}}$ and A is coloured with k which finishes the proof. It suffice to show that whenever $k_n = k$ and t_{n+1} is constructed then condition (3) is satisfied. Fix an $n \in \omega$ such that $k_n = k$. Consider the set

$$R_n = \{t \in T \cap M : \max(t) \geq \delta_n, t \cap (\varepsilon_n + 1) = t_n, \{t_i : i \leq n\} \subseteq C_t \text{ and } f(t) = k_n\}$$

and we wish to show that $R_n \neq \emptyset$.

It suffices to show that

$$R = \{t \in T : \max(t) \geq \delta_n, t \cap (\varepsilon_n + 1) = t_n, \{t_i : i \leq n\} \subseteq C_t \text{ and } f(t) = k_n\}$$

is not empty as $R_n = R \cap M$ and $R \in M$. However $t \in R$ which finishes the proof. $\square$

6.7 Open Problems

First, we mention that the following most general form of the Erdős-Hajnal problem is still open:

Problem 6.7.1. *Does every uncountably chromatic graph contain an ω-connected subset?*

We know, by Theorem 6.3.5, that this ω-connected set can only be countable in some cases however excluding countable ω-connected subsets seems to be a very hard problem. Certainly, our construction in Theorem 6.3.5 contains several countably infinite complete subgraphs (as shown in Proposition 6.6.1).

We don't know how essential it is to consider trees of the form $T(S)$ in finding uncountably chromatic ladder subgraphs. In particular:

Problem 6.7.2. *Suppose that T is a non special tree without uncountable chains. Is there a ladder system $\underline{C}$ on T such that the subgraph $X_{\underline{C}}$ of $G(T)$ is uncountably chromatic?*

One might start by looking at $\sigma\mathbb{Q}$ and Souslin trees first. In general about Hajnal-Máté graphs on ω_1, we make the following conjecture:

Conjecture 6.7.3. *It is consistent that $\clubsuit$ holds while there are no Hajnal-Máté graphs on ω_1.*

A recent (and long awaited) result of S. Shelah and H. Mildenberger [91] is the consistency of $\clubsuit$ with "every Aronszajn tree is special". We believe that their method can provide a positive solution to our conjecture.

We know that the trees $T(S)$ are quite rigid (see Theorem 1.5.5). Is this true for uncountably chromatic subgraphs? In particular:

Problem 6.7.4. *Are there disjoint stationary sets $S_0, S_1 \subseteq \omega_1$ such that $G(T(S_0))$and $G(T(S_1))$ has a common uncountably chromatic subgraph? If yes, can this subgraph be defined by ladder systems on $T(S_i)$?*

Also, there are several natural directions in which research can be continued on trees and ladder systems. In particular:

- finding applications in general topology using our framework,
- investigating minimal walks on trees along ladder systems.

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