On spaces with small dense sets

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Introduction

Goal: study topological spaces X which have a **dense set** $\bigcup\{D_n : n \in \omega\}$ so that D_n is *small*.

- separable spaces: D_n can be chosen singleton/finite/countable.
- d-separable spaces: D_n can be chosen discrete;
- e-separable spaces: D_n can be chosen closed and discrete;

- how do products/powers behave?
- study related cardinal functions.

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Separable spaces and products

Recall: X is separable iff there is a countable dense subset of X i.e. $d(X) = \aleph_0$.

If X is separable then X has a basis of size $\leq \mathfrak{c} = 2^{\aleph_0}$ and $|X| \leq 2^{\mathfrak{c}}$.

[Pondiczery 1944] If X is separable then $X^{\mathfrak{c}}$ is separable but $X^{\mathfrak{c}^+}$ is not.

- fix a countable dense D in X,
- $\mathfrak{c} = |\mathbb{R}|$, let $\mathcal{Q} \subseteq \mathcal{P}(\mathfrak{c})$ correspond to rational intervals in $\mathfrak{c}$,
- let $f \in E$ iff $f \in X^{\mathfrak{c}}$ and there are $\{I_k : k < m\}$ from $\mathcal{Q}$ and $d, d_k \in D$ so that $f \restriction I_k = d_k$ and $f \restriction \mathfrak{c} \setminus \cup_{k<m} I_k = d$;
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[Kurepa 1936] X is d-separable iff there is a σ-discrete dense subset of X.

- introduced as property K_0 in his study of Suslin's problem .

If X is separable or metrizable then X is d-separable.

- **[K 1936]** a Suslin-continuum is not d-separable.
  - size of discrete sets $\leq \aleph_0 < \aleph_1 \leq$ size of dense sets.
- **[Todorcevic 1981]**
  - In ZFC, there is a non d-separable continuum.
  - Consistently, any LOTS of density $\aleph_1$ is d-separable.

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Products and d-separable spaces

Recall: $X^{\mathfrak{c}}$ is separable if X is separable.

[Arhangelskii 1981] Any product of d-separable spaces is d-separable.

[Juhász, Szentmiklóssy 2008] X^κ is d-separable if there is a discrete subset of X^κ of size $d(X)$.

Note: $\{x \in 2^\kappa : |x^{-1}(1)| = n\} \subseteq D(2)^\kappa$ is discrete for any $n \in \omega$.

$\Rightarrow D(\kappa) \hookrightarrow D(2)^\kappa \hookrightarrow X^\kappa$.

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Compact spaces and powers

[JS 2008] X^ω is d-separable for any compact X.

- $(\omega^*)^n$ is not d-separable for any $n \in \omega$.

[JS 2008] X^2 has a discrete subset of size $d(X)$ for any compact X.

- Wlog: any non empty open subset has weight $w(X)$ (not trivial).
- find $(x_\alpha, y_\alpha) \in U_\alpha \times V_\alpha \subseteq X^2$ so that $U_\alpha \cap V_\alpha = \emptyset$ for $\alpha < d(X)$,
- there is a open $H \neq \emptyset$ so that $K = \overline{H} \subseteq X \setminus \overline{\{x_\alpha, y_\alpha : \alpha < \beta\}}$,
- $\{U_\alpha, V_\alpha : \alpha < \beta\}$ generate a coarser topology on K than the original compact so cannot be Hausdorff.
- Let $x_\beta, y_\beta \in K$ witness this; then $(x_\beta, y_\beta) \notin U_\alpha \times V_\alpha$ for $\alpha < \beta$.
- Now take any disjoint open U_β, V_β with $(x_\beta, y_\beta) \in U_\beta \times V_\beta \subseteq H^2$.

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- Now take any disjoint open U_β, V_β with $(x_\beta, y_\beta) \in U_\beta \times V_\beta \subseteq H^2$.

Compact spaces and powers

[JS 2008] X^ω is d-separable for any compact X.

- $(\omega^*)^n$ is not d-separable for any $n \in \omega$.

[JS 2008] X^2 has a discrete subset of size $d(X)$ for any compact X.

- Wlog: any non empty **open subset has weight $w(X)$** (not trivial).
- find $(x_\alpha, y_\alpha) \in U_\alpha \times V_\alpha \subseteq X^2$ so that $U_\alpha \cap V_\alpha = \emptyset$ for $\alpha < d(X)$,
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Some open problems

[JS 2008] Is there a **compact** space X in ZFC so that X has **no discrete subsets of size $d(X)$**?

- a compact L-space (e.g. a Suslin-continuum) is a consistent example.

[Burke, Tkachuk 2013] CH + $\diamondsuit(S^{\omega_2}_{\omega_1})$ implies that X^ω is not d-separable for some countably compact X.

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[Kurepa 1936] X is e-separable iff there is a σ-closed discrete dense subset of X.

- Kurepa named these K_0'-spaces.

If X is separable or metrizable then X is e-separable.

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Powers and large closed discrete sets

- Products of d-separable are d-separable.
- X^κ is d-separable if it has a discrete subset of size $d(X)$.

[Alas] If X is e-separable then X^κ is e-separable for all $\kappa \leq \mathfrak{c}$.

X^κ is e-separable if there is a closed discrete subset of X^κ of size $d(X^\kappa)$.

- Suppose that $\{d_\xi\}_{\xi<\delta} \subseteq X^I$ is dense in X^I and $\{e_\xi\}_{\xi<\delta} \subseteq X^{\kappa \setminus I}$ is closed discrete in $X^{\kappa \setminus I}$.
- $\{d_\xi \cup e_\xi\}_{\xi<\delta} \subseteq X^\kappa$ **is closed discrete in** X^κ.
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Are there large closed discrete sets?

Recall: X^κ is e-separable if there is a closed discrete subset of size $d(X^\kappa)$.

[Mycielski 1964] $D(\omega)^\kappa$ contains a closed discrete set of size κ for every κ less than the 1^{st} weakly inaccessible cardinal.

This helps if X **has an infinite closed discrete subset** i.e. not countably compact.

Suppose that X is not countably compact. Then $X^{d(X)}$ is e-separable if $d(X) <$ the 1^{st} weakly inaccessible cardinal.

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[Lós 1959] $D(\omega)^{2^\kappa}$ contains a closed discrete set of size κ for every κ less than the 1^{st} measurable cardinal.

Suppose that X is not countably compact. Then $X^{2^{d(X)}}$ is e-separable if $d(X) <$ the 1^{st} measurable cardinal.

What happens at a measurable?

If $\kappa > \omega$ is measurable then $D(\omega)^\kappa$ has no closed discrete subsets of size κ so is not e-separable.

Note: $D(\omega)^\kappa$ has closed discrete sets of all sizes $< \kappa$.

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Products - two reduction steps

Recall: $X^{\mathfrak{c}}$ is e-separable if X is e-separable.

Focus: products of $\kappa \leq \mathfrak{c}$ terms b.c. $D(2)^{\mathfrak{c}^+}$ is not e-separable.

Suppose that $\kappa \leq \mathfrak{c}$. Then the following are equivalent:

- every product of at most κ many *e*-separable spaces is *e*-separable;
- every product of at most κ many discrete spaces is *e*-separable.

$\Rightarrow$ **finite products preserve e-separability**.

The product of at most κ many discrete spaces of cardinality at least κ is e-separable.

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[Mrowka 1970] Define $\kappa \notin \mathcal{M}^*$ iff any product of κ many discrete spaces $X = \prod\{X_\alpha : \alpha < \kappa\}$ with each of size $< \kappa$ has no closed discrete set of size κ.

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Suppose that $\kappa \leq \mathfrak{c}$ is minimal so that there is a family of κ e-separable spaces with non e-separable product. Then $\kappa \notin \mathcal{M}^*$.

The preservation theorem:

Suppose there are no weakly inaccessible cardinals $\leq \mathfrak{c}$. Then the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable.

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Suppose there are no weakly inaccessible cardinals $\leq \mathfrak{c}$. Then the product of at most $\mathfrak{c}$ many e-separable spaces is e-separable.

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$\mathcal{M}^*$ and weak compactness

$\mathcal{L}_{\kappa,\omega}$ is the infinitary language which allows conjunctions and disjunctions of $< \kappa$ formulas and universal or existential quantification over finitely many variables.

$\mathcal{L}_{\kappa,\omega}$ is *weakly compact* iff every set of at most κ sentences Σ from $\mathcal{L}_{\kappa,\omega}$ has a model provided that every $S \in [\Sigma]^{<\kappa}$ has a model.

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[Erdős, Tarski 1961] κ is a weakly compact cardinal iff $\kappa \to (\kappa)^2_2$.

κ is a weakly compact cardinal iff it is strongly inaccessible and $\mathcal{L}_{\kappa,\omega}$ is weakly compact.

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Some cardinal functions

Recall some classical cardinal functions:

- **density**: $d(X) = \min\{|D| : D \subseteq X \text{ is dense in } X\}$;
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If X is d-separable:

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There is a 0-dimensional e-separable space X such that

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The plan is that $X = Y \cup E \subseteq 2^{2^{\mathfrak{c}}}$, both Y, E are dense and

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- [Moore 2006] there is a dense $Y \subseteq \omega^{\omega_1}$ such that any σ-discrete is nowhere dense.
- if $\omega_2 \leq \mathfrak{c}$ then there is a countable dense $D \subseteq 2^{\omega_2}$,
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Does $d(X) = d_s(X)$ **for compact**, d-separable X?

[Moore 2008] There is an L-space X such that X^2 is d-separable.

[Peng 2015] There is an L-space X such that X^2 is e-separable.

Note: $\aleph_0 = s(X) = e(X) < d(X) = \aleph_1$ so X is not d-separable.

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Thank you for your attention!