Ladder system uniformization on trees

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Supported in part by FWF Grant I1921, OTKA 113047.

Introduction to uniformization

A **ladder system on ω_1** is $\mathbf{C} = (C_\alpha)_{\alpha \in \lim \omega_1}$ so that C_α is a cofinal subset of α in order type ω. **A colouring of C** is $\mathbf{f} = (f_\alpha)_{\alpha \in \lim \omega_1}$ so that $f_\alpha : C_\alpha \to \omega$.

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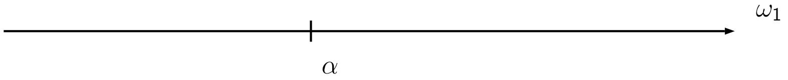

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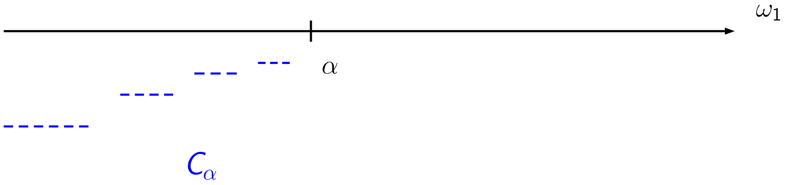

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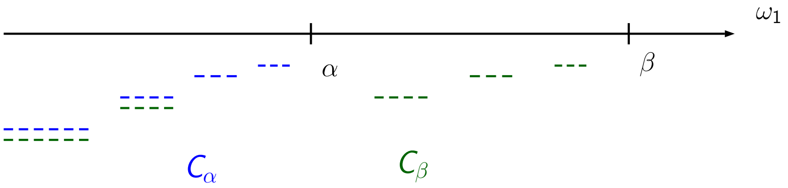

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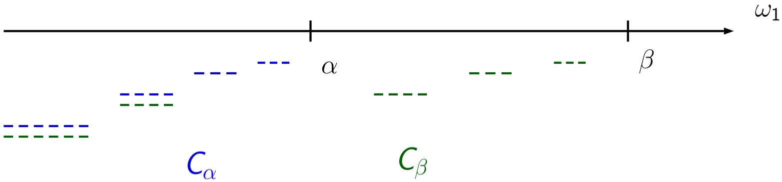

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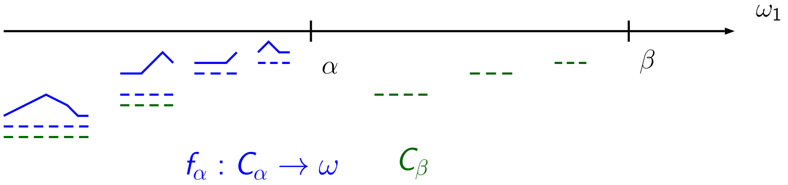

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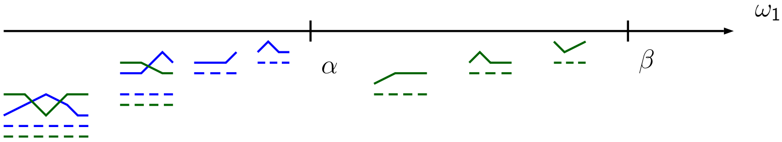

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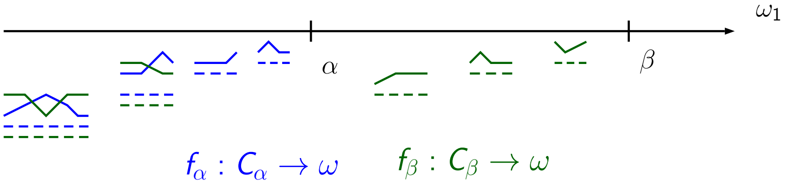

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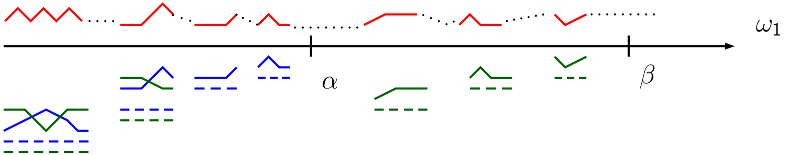

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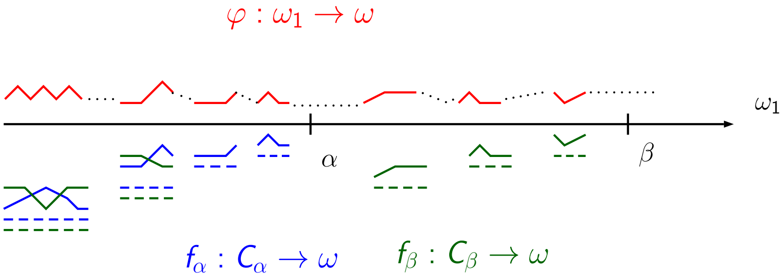

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$MA_{\aleph_1}$ implies that any ladder system colouring has an ω_1-uniformization.

$2^{\aleph_0} < 2^{\aleph_1}$ implies that any ladder system has a monochromatic 2-colouring without ω_1-uniformization.

The motivation to study these objects come from the Whitehead-and related algebraic problems, various topological questions (e.g. normal Moore-space conjecture), the study of forcing axioms that allow CH.

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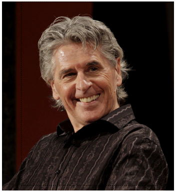

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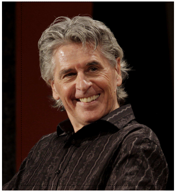

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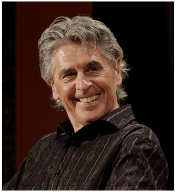

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A few words on trees

We will look at $\boxed{\text{pruned trees } T \text{ of height } \omega_1.}$

- ω_1 itself is a tree of height ω_1;
- **Aronszajn-trees**: all the levels and chains are countable;
- **Suslin-trees**: all the antichains and chains are countable;
- $\sigma\mathbb{Q}$, the set of all **well ordered $t \subset \mathbb{Q}$ which have a maximum** with the initial segment relation;
  - its levels have size continuum, but chains are countable;
  - special: the union of countably many antichains.

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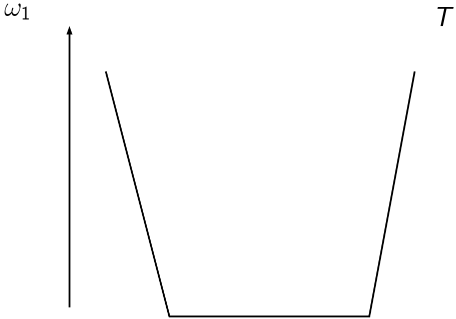

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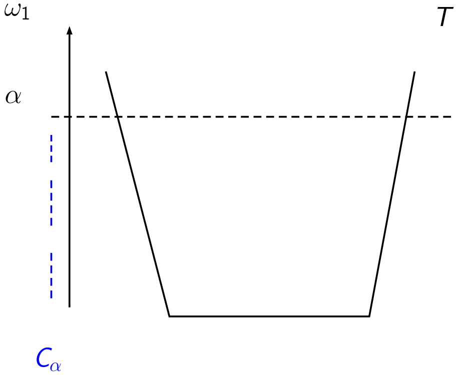

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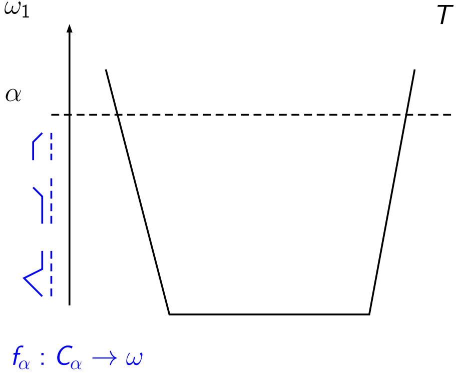

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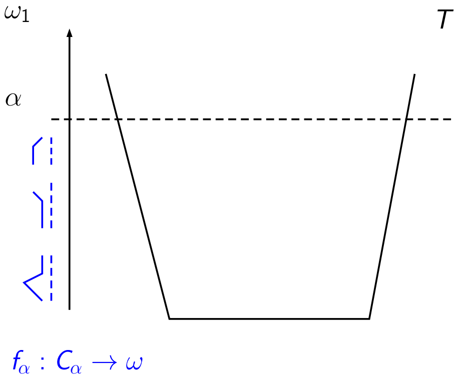

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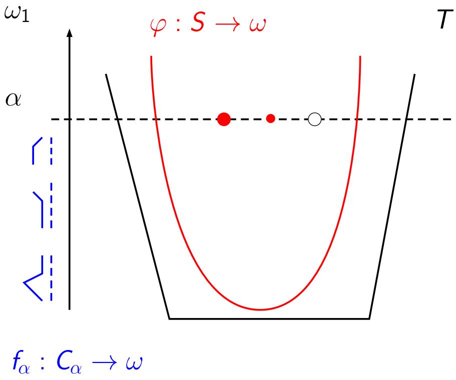

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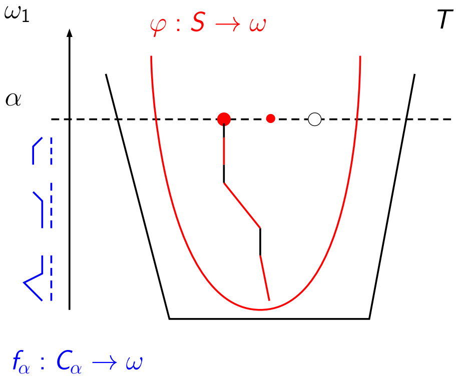

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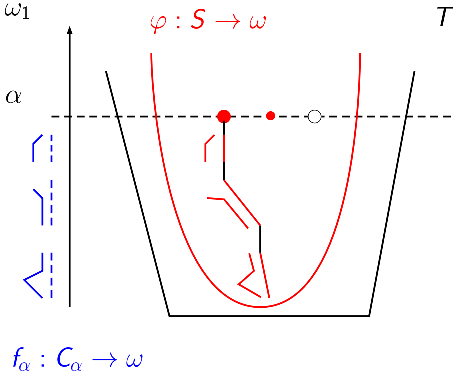

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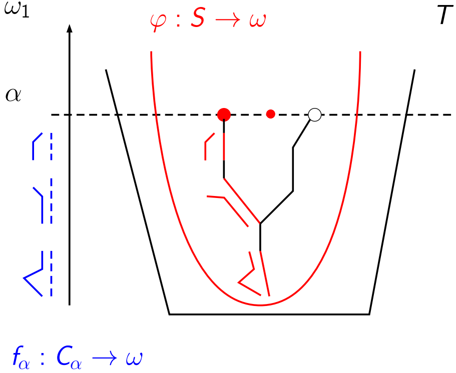

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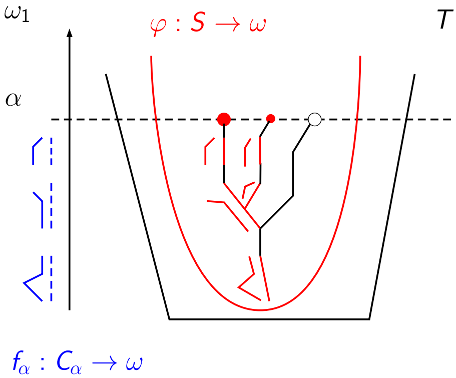

Uniformization and minimal linear orders

A linear order L is minimal if L embeds into all its suborders of size $|L|$.

- the only minimal linear orders of size $\aleph_0$ are $\pm\omega$;
- under PFA, **Baumgartner:** any $\aleph_1$-dense set of reals is minimal and **Todorcevic:** there are minimal A-lines;
- **Baumgartner:** under $\diamondsuit^+$, there are minimal A-lines.

Consistently, the only minimal linear orders of size $\aleph_1$ are $\pm\omega_1$.

Consistently, CH holds and for any A-tree T, any ladder system colouring has a T-uniformization.

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- under PFA, **Baumgartner:** any $\aleph_1$-dense set of reals is minimal and
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Moore, 2007

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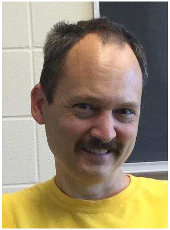

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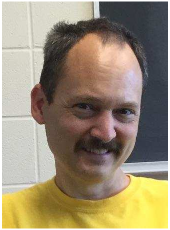

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Under $\diamondsuit^+$, there is a lexicographically ordered Suslin-tree which is a minimal linear order.

Baumgartner 1982/2017

Question: does a single Suslin-tree or $\diamondsuit$ suffice for the construction?

Consistently, there is a Suslin-tree and the only minimal linear orders of size $\aleph_1$ are $\pm\omega_1$.

D. Soukup 2018

- $(V = L)$ take a full Suslin-tree R,
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Non-uniformization results on trees

$2^{\aleph_0} < 2^{\aleph_1} \Rightarrow$ Any l.s. **C** has a monochrom. 2-colouring without ω_1-uniformization.

$2^{\aleph_0} < 2^{\aleph_1} \Rightarrow$ for **any Suslin-tree T and any C**, there is a monochrom. 2-colouring without T-uniformization.

$\diamondsuit \Rightarrow$ **any A-tree T there is a C** and a monochrom. 2-colouring without T-uniformization.

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$\diamondsuit \Rightarrow$ any A-tree T and l.s. $\mathbf{C}$ there is a 2-colouring without T-uniformization.

$\diamondsuit^+$ implies that for any ladder system $\mathbf{C}$, there is an A-tree T so that any monochromatic colouring of $\mathbf{C}$ has a T-uniformization.

Without any extra assumptions (in ZFC):

There is a map $\varphi : \sigma\mathbb{Q} \to \omega$ so that for any ladder system colouring $\mathbf{f}$ there is an A-tree $T \subseteq \sigma\mathbb{Q}$ so that $\varphi \restriction T$ is a T-uniformization of $\mathbf{f}$.

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Thank you very much! Any questions?

James E. Baumgartner

March 23, 1943 – December 28, 2011

For more details and open problems:

D. T. Soukup, **Ladder system uniformization on trees I & II**, prerint, **arXiv: 1806.03867**

D. T. Soukup, **A model with Suslin trees but no minimal uncountable linear orders other than ω_1 and $-\omega_1$**, submitted to the Israel Journal of Math., **arXiv: 1803.03583**.