High Davies-trees in infinite combinatorics

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Introduction

We explore a general construction scheme based on elementary submodels, to build arbitrary large structures by piecing together small local approximations.

- paradoxical sets and decompositions of the plane, and
- Bernstein-decompositions of arbitrary topological spaces.

Based on

"Infinite combinatorics plain and simple" [ArXiv: 1705.06195]

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Motivations - paradoxical covers

CH is the statement that $\mathfrak{c} = |\mathbb{R}| = \aleph_1$.

[Sierpinski 1919]
CH holds iff $\mathbb{R}^2 = S_0 \cup S_1$ so that

- S_0 has countable vertical segments, and
- S_1 has countable horizontal segments.

$A \subset \mathbb{R}^2$ is **a cloud** if there is some $a \in \mathbb{R}^2$ so that $\ell \cap A$ is finite for any line ℓ through a. Note that **two clouds cannot cover $\mathbb{R}^2$.**

[Komjáth, Schmerl 2001/2003] CH iff $\mathbb{R}^2$ is covered by three clouds,

$$\boxed{\mathfrak{c} \leq \aleph_n \quad \text{iff} \quad \mathbb{R}^2 \text{ is covered by } n+2 \text{ clouds.}}$$

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[Mazurkiewicz 1914] There is a set in $\mathbb{R}^2$ that meets each line in exactly two points.

- build by an induction of length $\mathfrak{c}$,
- **a partial 2-point set (of size $< \mathfrak{c}$) can be extended** to meet a given line in exactly two points.

[Jackson, Mauldin 2002] There is a set in $\mathbb{R}^2$ that meets each isometric copy of $\mathbb{Z}^2$ in exactly one point.

- **there are finite partial Steinhaus sets which cannot be extended**,
- the proof combines elementary number theory and mechanics to solve a countable approximation of the problem,
- to lift the countable case and piece together a full Steinhaus set a transfinite induction using elementary submodels .

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We say $X_0 \sqcup X_1$ is a **Bernstein-decomposition** of a space X if there are no copies of the Cantor set in either X_0 or X_1.

[Bernstein 1908] Any topological space of size $\leq \mathfrak{c}$ admits a Bernstein-decomposition.

- X has $\leq \mathfrak{c}$-many Cantor subspaces, go through them by an induction of length $\mathfrak{c}$,
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> [W. Weiss 1980] If $V = L$ then any T_2 space has a Bernstein-decomposition.

[Shelah 2004] Using a supercompact, consistently there is a 0-dim, T_2 space X of size $\aleph_{\omega+1}$ without Bernstein-decomposition.

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Elementary submodels - the basics

$(V, \in)$ is the set-theoretic universe.

- $H(\Theta)$ is the family of all sets of hereditary cardinality $< \Theta$,
- the larger Θ is, the more $(H(\Theta), \in)$ resembles V,
- $M \subset H(\Theta)$ is an **elementary submodel** if for any first-order formula ϕ with parameters in M, $H(\Theta) \models \phi$ if and only if $M \models \phi$.

The downward Löwenhein-Skolem theorem says that for any countable $x \subset H(\Theta)$, **there are countable elementary $M \prec H(\Theta)$ so that $x \subset M$.**

1. M is **closed under operations** defined using parameters in M,
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- $\omega \subset M$ so for each $n \in \omega$, $f(n) \in M$ as well; so $A = \mathrm{ran}(f) \subset M$.

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Elementary submodels - a simple application

M is closed under operations defined using parameters in M,
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If A is an uncountable family of finite sets then there is a finite set r and uncountable $B \subset A$ so that $a \neq b \in B$ **implies that** $r = a \cap b$.

- take a countable elementary $M \prec H(\Theta)$ such that $A \in M$;
- there is some $a \in A \setminus M$ because A is uncountable, let $r = a \cap M$;
- take a **maximal** $B \subset A$, $B \in M$, $r \subseteq b$ if $b \in B$ and $\{b \setminus r : b \in B\}$ **is pairwise disjoint**;
- B is uncountable, otherwise $B \subset M$ but then $B \cup \{a\}$ **is a strictly larger** set with the above properties.

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Elementary submodels - a simple application

M is closed under operations defined using parameters in M,
if M is countable, $A \in M$ then either $A \subset M$ or A is uncountable.

If A is an uncountable family of finite sets then there is a finite set r and uncountable $B \subset A$ so that $a \neq b \in B$ **implies that** $r = a \cap b$.

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Chains of countable elementary submodels

A standard trick in infinite combinatorics is to use filtrations:
an increasing chain of small elementary submodels to cover a large set.

For any set $X \in H(\Theta)$ of size $\aleph_1$, there is a (continuous) **increasing chain $(M_\alpha)_{\alpha<\omega_1}$ of countable elementary submodels** of $H(\Theta)$ so that

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People always talk about the weather

CH implies that $\mathbb{R}^2$ is the union of three clouds.

Fix any three non-collinear points $a_0, a_1, a_2 \in \mathbb{R}^2$,

- let $\mathcal{L}^k$ denote the lines through a_k, $\mathcal{L} = \cup \mathcal{L}^k$, $\mathcal{L}'$ the three lines determined by pairs of $\{a_k\}$.

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[Davies, 1963] Take any set of size $\mathfrak{c}$ and cover with $M_\emptyset$ of size $\mathfrak{c}$.

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- repeat until all terminal models are countable.

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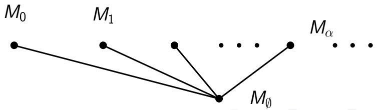

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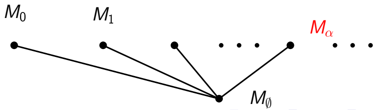

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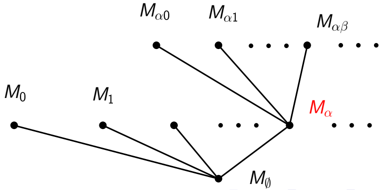

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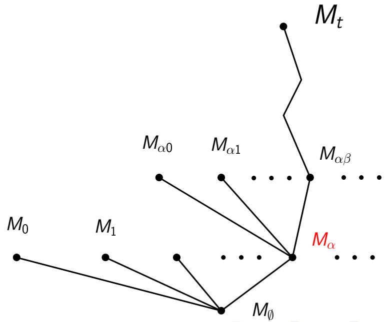

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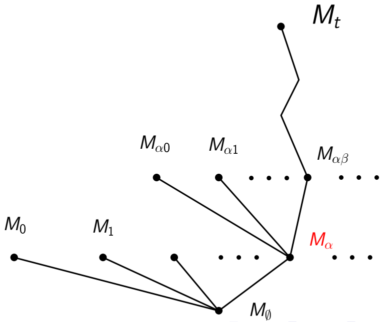

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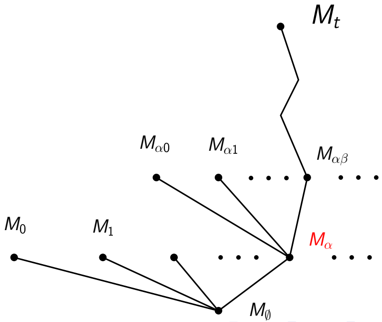

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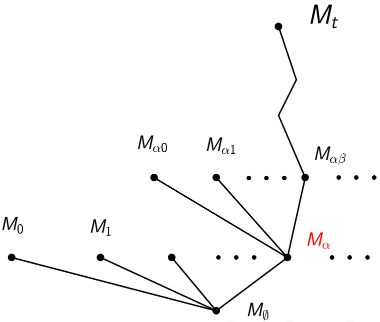

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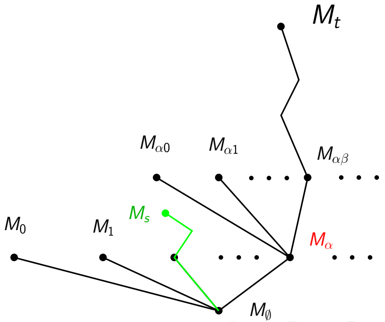

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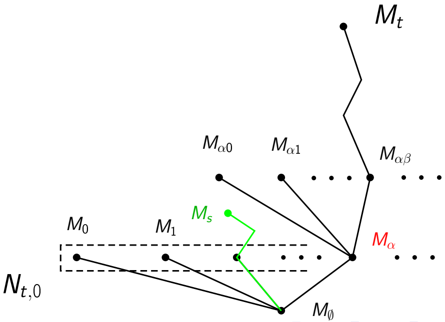

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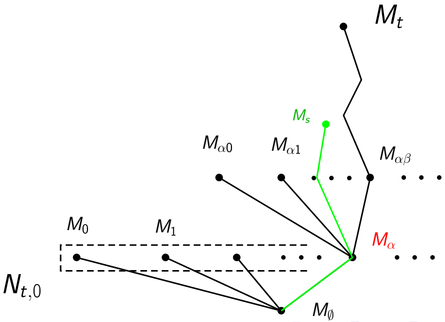

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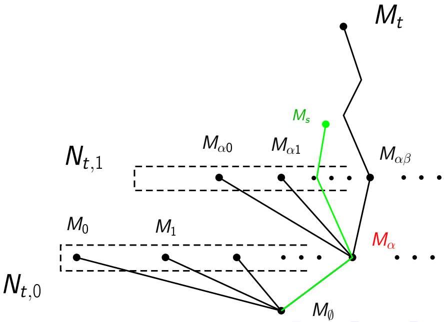

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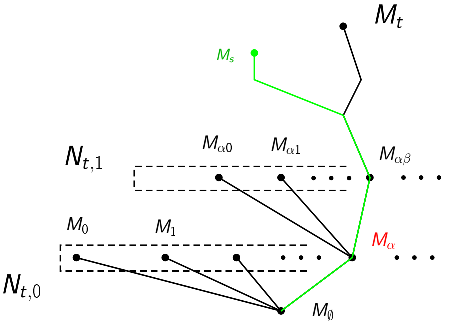

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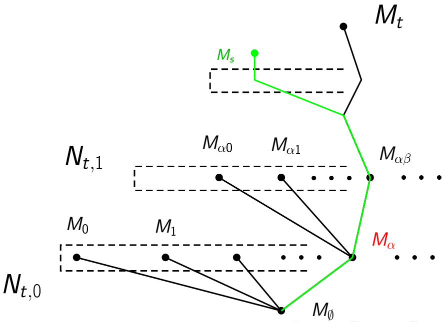

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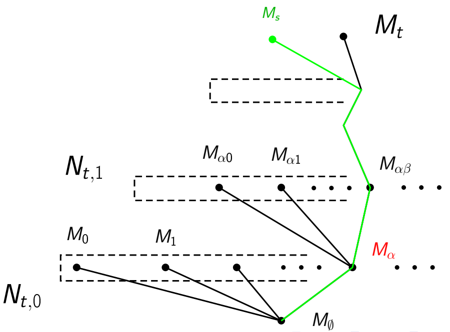

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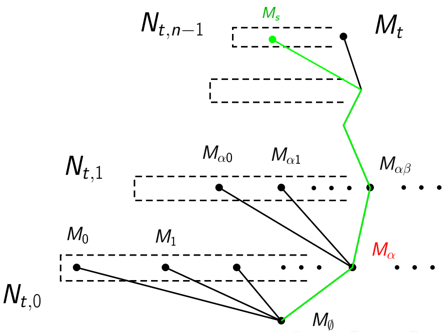

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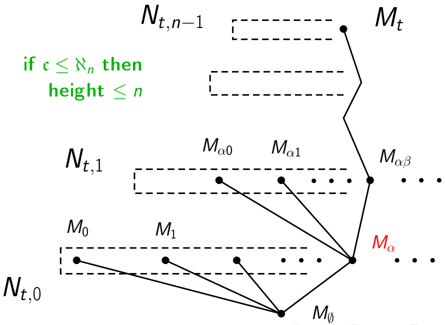

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Suppose that κ is cardinal, x is a set. Then there is $\kappa << \theta$ and a sequence $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $H(\theta)$ so that

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$\mathfrak{c} \leq \aleph_n$ implies there is a cover by $n+2$ clouds

Take $n+2$ points $a_0, \ldots, a_{n+1}$ in general position, and a **Davies-tree** $(M_\alpha)_{\alpha<\mathfrak{c}}$ covering $\mathbb{R}^2$ and $\mathcal{L}$, containing the points a_k, and so that

$M_{<\alpha}$ **is the union of** $\leq n$ **elementary submodels**.

$\mathcal{L}_\alpha$ those lines $\ell \in \mathcal{L} \setminus \mathcal{L}'$ that appear first in M_α,
$F(\ell_{\alpha j}) = \bigcup\{\ell \cap \ell_{\alpha j} : \ell \in \mathcal{L}' \cup \{\ell_{\alpha i} : i < j\}\}$,
$A_k = \{a_k\} \cup \bigcup\{F(\ell) : \ell \in \mathcal{L}^k\}$.

Any $x \in \mathbb{R}^2$ is covered by $A_0, \ldots, A_{n+1}$.

- there is α so that x **appears in** M_α **first**, we suppose $x \notin \mathcal{L}'$,

  at most n of the $n+2$ lines $\ell[x, a_0], \ldots, \ell[x, a_{n+1}] \in M_\alpha$ could appear in previous models

- $M_{<\alpha} = \bigcup_{k<n} N_{\alpha k}$ and if $\ell[x, a_i], \ell[x, a_j] \in N_{\alpha k}$ then

$$\{x\} = \ell[x, a_i] \cap \ell[x, a_j] \subset N_{\alpha k} \subset M_{<\alpha}$$

- $\ell[x, a_0], \ell[x, a_1] \in \mathcal{L}_\alpha$ and there is $i < j$ such that $\ell_{\alpha i} = \ell[x, a_0]$ and $\ell_{\alpha j} = \ell[x, a_1]$,
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Countable models $\to$ enumeration in type ω,
$\to$ deal with finite pieces one at a time.

M is countably closed if $x \subseteq M$, $|x| \leq \omega$ implies $x \in M$.

- there are countably closed $M \prec H(\theta)$ of size $\mathfrak{c}$;

How are these models helpful?

- they can cover larger spaces and the **models can talk about countably infinite subsets**,
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High Davies-trees

We say that a **high Davies-tree for κ over x** is a sequence $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $H(\theta)$ for some large enough regular θ such that

(I) $[M_\alpha]^\omega \subset M_\alpha$, $|M_\alpha| = \mathfrak{c}$ and $x \in M_\alpha$ for all $\alpha < \kappa$,

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Remark: **no high Davies-trees for $\kappa \geq \aleph_\omega$** if $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

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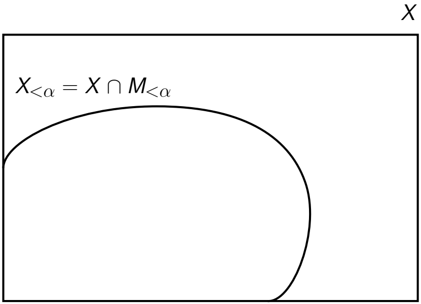

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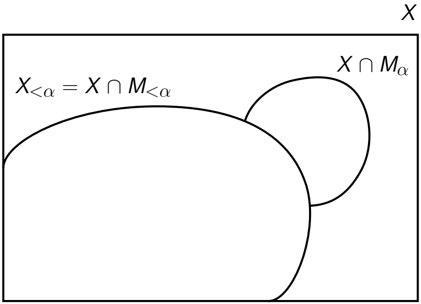

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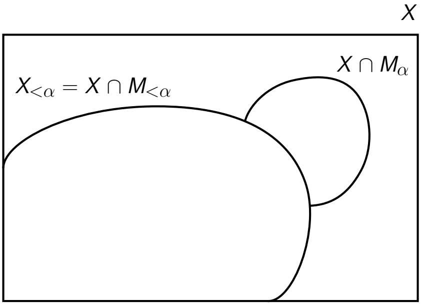

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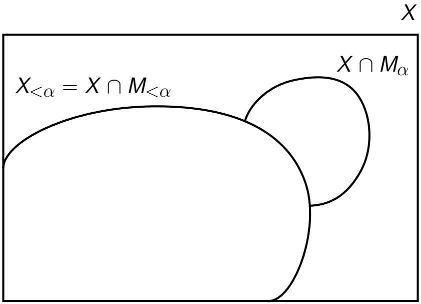

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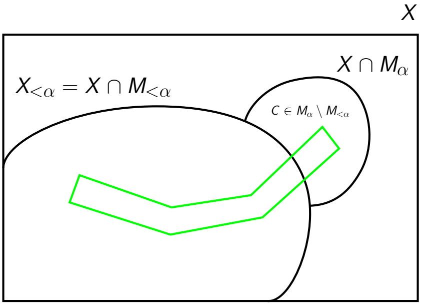

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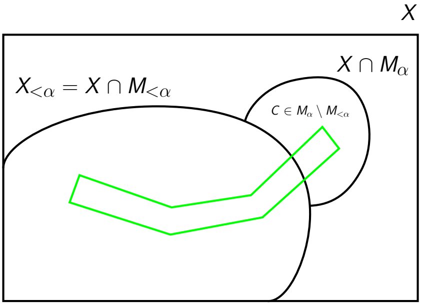

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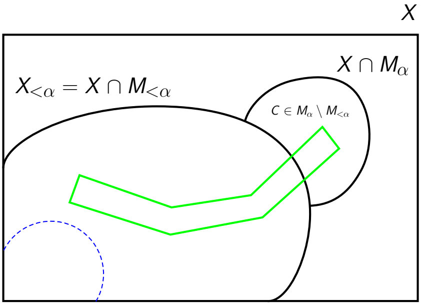

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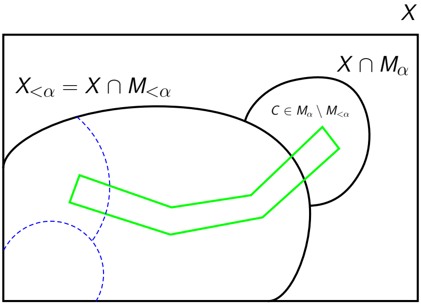

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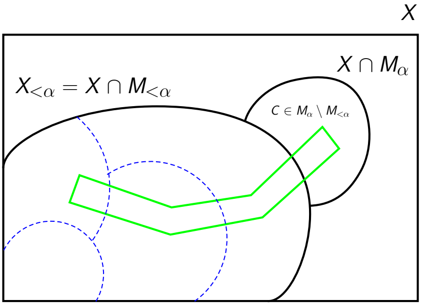

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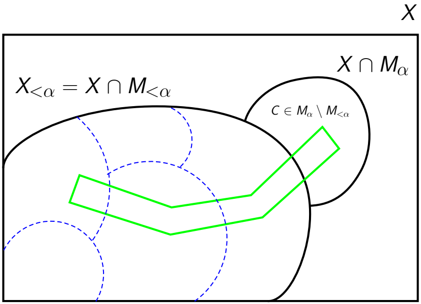

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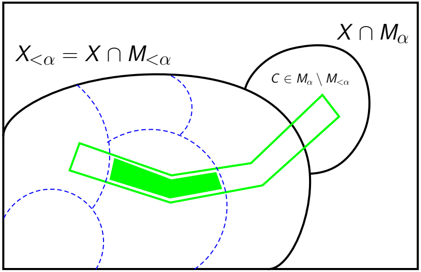

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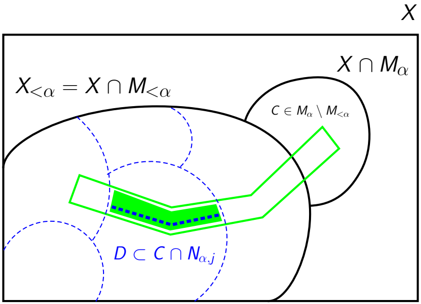

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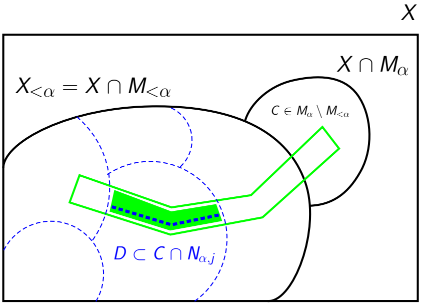

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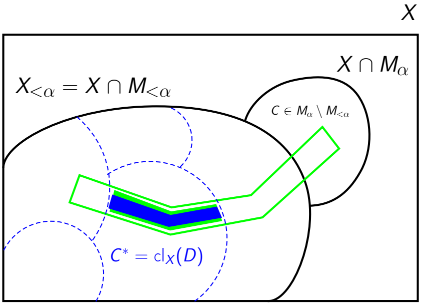

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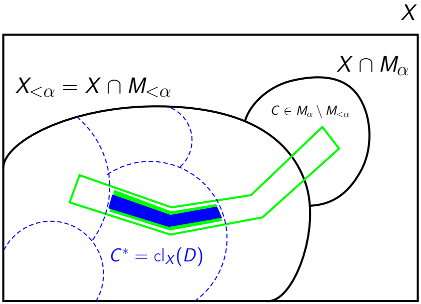

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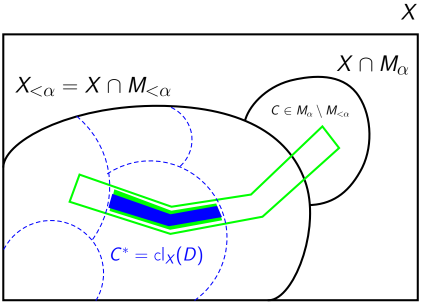

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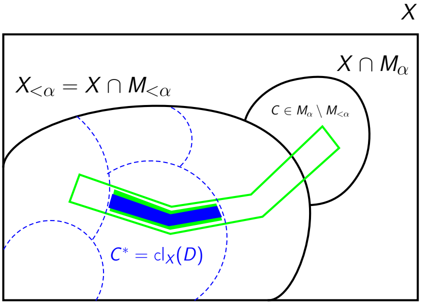

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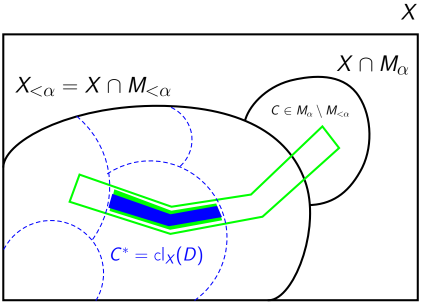

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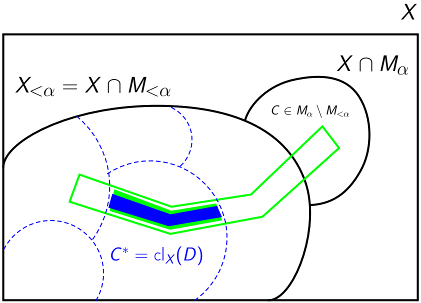

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Some open problems

[A. Miller 1989]
(V=L) There is a co-analytic 2-point set in $\mathbb{R}^2$.

- [Sierpinski] Is there a Borel 2-point set?
- It can never be F_σ, but how about G_δ?

[Gardner, Mauldin 1988] (CH) For $n \geq 3$, there is a bijection $f : \mathbb{R}^n \to \mathbb{R}^n$ which maps each circle onto a countable union of line segments.

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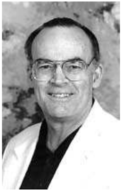

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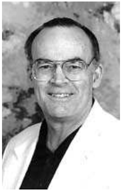

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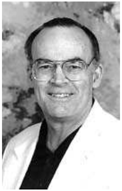

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