# Chromatic number - finite, infinite and uncountable

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Goal: to present some fundamental results and open problems on **chromatic number of finite/infinite graphs**.

- first organized effort: P. Erdős and A. Hajnal in the 1960s;
- significant contributions: P. Komjáth, S. Todorcevic, S. Shelah, C. Thomassen...
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**Definition**

*The **chromatic number** of a graph $G$, denoted by $Chr(G)$, is the least cardinal $\kappa$ such that **the vertices of $G$ can be covered by $\kappa$ many independent sets**.*

**[Galvin, Komjáth 1991]**:
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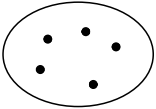

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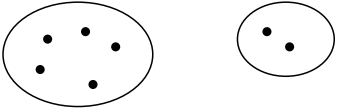

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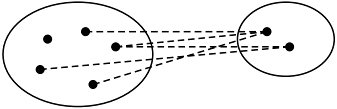

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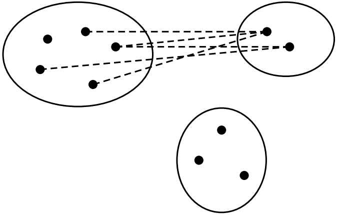

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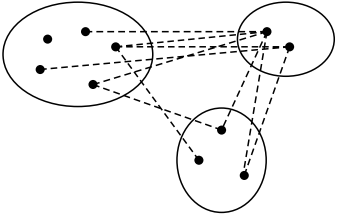

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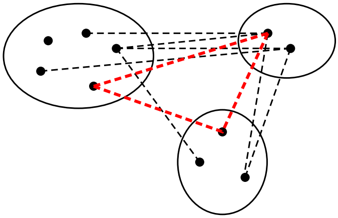

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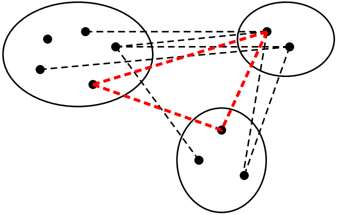

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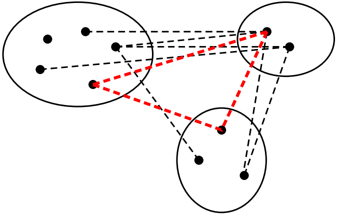

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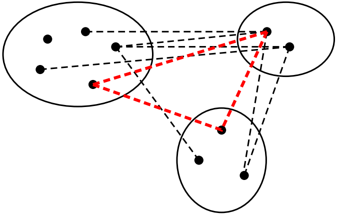

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# Finite vs infinite vs uncountable

- **de Bruijn-Erdős compactness 1951:** If $n \in \mathbb{N}$ and $\text{Chr}(H) \leq n$ **for any finite** $H \subseteq G$ then $\text{Chr}(G) \leq n$.

Is there a similar reflection for uncountable chromatic numbers?

- **Komjáth 1988:** Consistently, there is $G$ with $\text{Chr}(G) = \aleph_2$ **no subgraph** $H \subseteq G$ **with** $\text{Chr}(H) = \aleph_1$.

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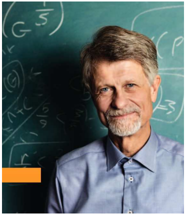

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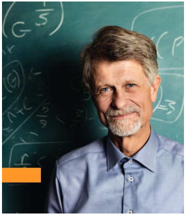

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# Shift graphs

Define the **shift graph** $\mathrm{Sh}_n(\kappa)$ on $[\kappa]^n$ by connecting

$$u = \{\xi_0 < \cdots < \xi_{n-1}\} \text{ with } v = \{\xi_1 < \cdots < \xi_n\}.$$

If $\kappa$ is large enough then **the chromatic number of $\mathrm{Sh}_n(\kappa)$ is large**.

**No odd cycles** of length $\leq 2n - 1$.

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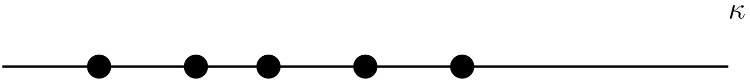


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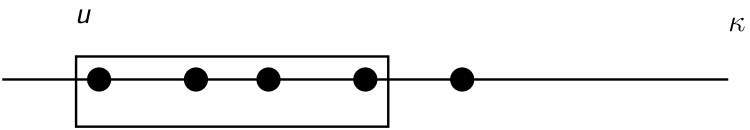


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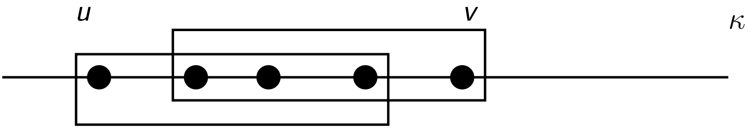


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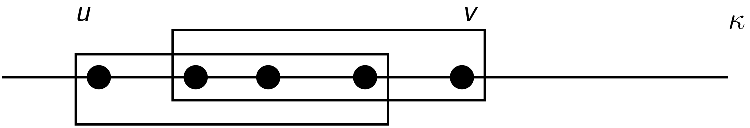


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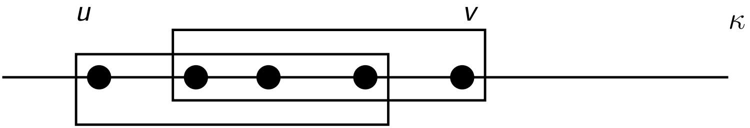


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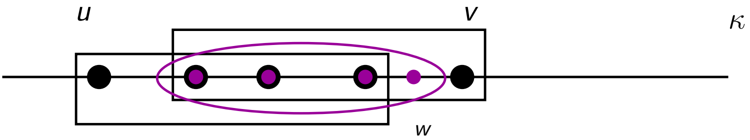


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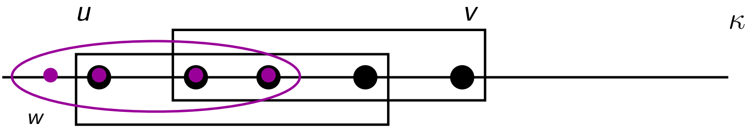


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- Recall: if $\mathrm{Chr}(G) > \aleph_0$ then $G$ contains all odd cycles of sufficiently large length.
- If $\mathrm{Chr}(G), \mathrm{Chr}(H) > \aleph_0$ then they have a common odd cycle i.e. a common subgraph of chromatic number 3.

[Erdős] Suppose that $\mathrm{Chr}(G), \mathrm{Chr}(H) > \aleph_0$.
Is there a common 4-chromatic subgraph of $G$ and $H$?

No legitimate guess on what these subgraphs could be;
no partial/consistency results known...

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**[Erdős]** Suppose that $\mathrm{Chr}(G), \mathrm{Chr}(H) > \aleph_0$.
**Is there a common 4-chromatic subgraph of $G$ and $H$?**

No legitimate guess on what these subgraphs could be;
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# Open problems - a general theme

Suppose that $\mathcal{P}$ **is some property of graphs**.

**Examples**: $\mathcal{P}$ can express simply the size of $H$, or

- density e.g. large minimal degree, connectedness,
- sparseness e.g. avoiding certain (finite) subgraphs.

Given $\mathrm{Chr}(G) > \aleph_0$, **is there** $H \subseteq G$ **with** $\mathrm{Chr}(H) > \aleph_0$ **and property** $\mathcal{P}$**?**

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**Definition.** $H$ is $n$-connected iff $|H| \geq n$ and $H$ remains connected after the removal of $< n$ vertices.

**[Erdős-Hajnal, 1966, 1985...]** Does every graph $G$ with $\mathrm{Chr}(G) > \aleph_0$ **contain $\aleph_0$-connected subgraphs $H$ with $\mathrm{Chr}(H) > \aleph_0$**?

Following results by Komjáth:

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How ubiquitous are these objects?

**[Erdős, 1969, 1970s...]** Is there a map $f$ so that if $\mathrm{Chr}(G) > f(k)$ then there is a **$\Delta$-free $H \subseteq G$ with $\mathrm{Chr}(H) > k$**?

- **[Rödl, 1977]** Yes, for finite values of $k$.
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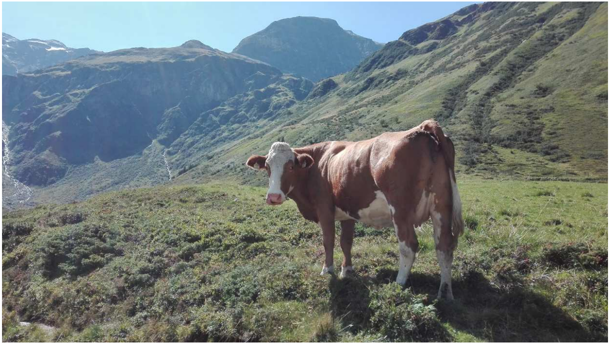
# Thank you for your attention!



Any questions?