Strongly surjective linear orders

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Introduction

Project goal: study uncountable linear orders L so that

$$K \hookrightarrow L \text{ if and only if } L \twoheadrightarrow K.$$

[R. Camerlo, R. Carroy and A. Marcone]

- disclaimer and introduction;
- various consistency results;
- is there an example in ZFC?
- open problems along the way.

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Consider the class of linear orders with order preserving embeddings.

Among countable linear orders:

- ω and $-\omega$ are the only **minimal** linear orders;
- $\mathbb{Q}$ is the **unique dense** l.o. without endpoints.

How about uncountable linear orders?

- ω_1 and $-\omega_1$ are minimal,

L is short if $\omega_1, -\omega_1 \not\hookrightarrow L$

- suborders of $\mathbb{R}$, or
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Strongly surjective linear orders

If $f : L \twoheadrightarrow K$ then $L = \sum_{k \in K} f^{-1}(k)$.

Select $\ell_k \in f^{-1}(k)$ and note that $K \simeq \{\ell_k : k \in K\} \hookrightarrow L$.

[CCM 2015] When is this implication reversible?

L is strongly surjective if $K \hookrightarrow L$ implies $L \twoheadrightarrow K$.

- $\omega, -\omega$ and $\mathbb{Q}$ are strongly surjective,
- strongly surjective $\Rightarrow$ short $\Rightarrow$ size $\leq 2^{\aleph_0}$.

If $L \subseteq \mathbb{R}$ is Borel and strongly surjective then $|L| \leq \omega$.

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Countable strongly surjective linear orders

We say that L is strongly surjective if $L \twoheadrightarrow K$ for any $K \hookrightarrow L$.

[CCM 2015] $\xi \in \mathrm{ORD}$ is strongly surjective iff $\xi = \omega^\alpha m$ where $\alpha < \omega_1$ and $m \in \omega$.

[CCM 2016] The set of countable, strongly surjective linear orders is the union of a Π^1_1-complete set (scattered ones) and $\sum^1_1$-complete set (non-scattered ones).

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Uncountable suborders of $\mathbb{R}$

[CCM 2016] If A is the unique κ-dense suborder of $\mathbb{R}$ (i.e. BA_κ holds) then A is strongly surjective.

[Baumgartner 1970] PFA $\rightarrow BA_{\aleph_1}$ [Neeman ?] Con($BA_{\aleph_2}$)

Note: these examples are all minimal and homogeneous under MA.

Consistently, there is an $\aleph_1$-dense, strongly surjective $L \subseteq \mathbb{R}$ which is not minimal and not homogeneous.

[Abraham, Rubin, Shelah 1985] Consistently, $MA_{\aleph_1}$ + OCA + ISA.

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Some questions

Does **every** uncountable, strongly surjective l.o. **contain a minimal suborder**?

$MA_{\aleph_1} \xrightarrow{?}$ there is an uncountable, strongly surjective $L \subseteq \mathbb{R}$

There is no 2-entangled, strongly surjective linear order.

- 2-entangled implies no minimal suborder, and
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Suppose $L \subseteq \mathbb{R}$ is **strongly surjective and rigid**. Is L countable?

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The effects of CH and relatives

CH implies $\neg \mathrm{BA}_{\aleph_1}$ and there are no minimal suborders of $\mathbb{R}$.

[CCM 2016] $2^{\aleph_0} < 2^{\kappa} \Rightarrow$ no strongly surjective $L \subseteq \mathbb{R}$ of size κ.

Every uncountable, strongly surjective linear order is Aronszajn if

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A Suslin example

($\diamondsuit^+$) There is a strongly surjective, lex. ordered Suslin-tree T.

Key property [CCM 2016]:

T is Suslin + doubly isomorphic to all large subtrees.

1. [Baumgartner 1982] the proof is oversimplified (false lemma);
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From [Moore 2007]:

- (A) $\equiv$ any ladder system colouring can be uniformized on an arbitrary Aronszajn tree;
- (CH + (A)) ω_1 and $-\omega_1$ are the only minimal uncountable l. orders;
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Open problems

Consistently, are there **strongly surjective** linear orders of **size** $> \aleph_2$?

[ARS 1985] Is it consistent that $\neg BA_{\aleph_1}$ but $A \hookrightarrow B$ **or** $B \hookrightarrow A$ for any two $\aleph_1$-dense $A, B \subseteq \mathbb{R}$?

Suppose that L is strongly surjective and $x \in L$. **Is $L \setminus \{x\}$ strongly surjective?**

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They are minimal under $MA_{\aleph_1}$.

Is the universal A-line η_C strongly surjective under PFA?

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Thank you for your attention!