Colouring large groups and monochromatic sumsets

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Ramsey theory: the study of unavoidable regularity

Positive partition relations: a large object, coloured with a small number of colours, always admits monochromatic subsets/substructures of relatively large size.

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> **Shelah 1995:** consistently, any function $f : \mathbb{R} \to \mathbb{R}$ is continuous on a nowhere meager set.

Negative partition relations: the existence of a colouring without large monochromatic substructures; paradoxical decompositions.

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Consistently, modulo some large cardinal,

if $f : \mathbb{R} \to r$ with $r \in \omega$ then there is an **infinite** $X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, Z. Vidnyánszky 2017

$X + X = \{x + y : x, y \in X\}$ i.e. repetitions are allowed.

- How does this fit into the theory (of partition relations)?
- Why allow repetitions and why only infinite?
- What goes into the proof of this result?

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Evolving partition relations

... an incomplete overview ...

Evolving partition relations

If $f : \omega \to r$ then there is an infinite $X \subset \omega$ with $f \restriction X$ constant.

P. H. Principle

$$\omega \to (\omega)^1_r$$

Evolving partition relations

If $f : [\omega]^k \to r$ then there is an infinite $X \subset \omega$ with $f \restriction [X]^k$ constant.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

Evolving partition relations

There is $f : [2^{\aleph_0}]^2 \to 2$ so that $f''[X]^2 = 2$ for any uncountable $X \subset 2^{\aleph_0}$.

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$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

W. Sierpinski, 1933

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Evolving partition relations

If $f : [\beth_{k-1}^{+}]^k \to r$ then $f \restriction [W]^k$ is constant for some uncountable $W \subseteq \beth_{k-1}^{+}$.

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$2^{\aleph_0} \not\to (\aleph_1)^2_2$

Erdős, Rado 1956

$\beth_{k-1}^{+} \to (\omega_1)^k_r$ for all $r < \omega$.

Evolving partition relations

$FS(X) = \{x_0 + x_1 + \dots + x_\ell : x_0 < \dots < x_\ell \in X\}$ i.e. no repetitions.

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$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Erdős, Rado 1956

$\beth^+_{k-1} \to (\omega_1)^k_r$ for all $r < \omega$.

N. Hindman, 1974

if $f : \mathbb{N} \to r$ then there is some infinite $X \subseteq \mathbb{N}$ so that $f \restriction FS(X)$ is constant.

Evolving partition relations

There is $f : [\aleph_1]^2 \to \aleph_1$ so that $f''[X]^2 = \aleph_1$ for any uncountable $X \subset \aleph_1$.

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S. Todorcevic, 1987

$$\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$$

Evolving partition relations

If $f : [2^{\aleph_0}]^2 \to 3$ then there is an uncountable $X \subset 2^{\aleph_0}$ with $|f''[X]^2| \le 2$.

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S. Shelah, 1988

$$\text{Con}(2^{\aleph_0} \to [\aleph_1]^2_3)$$

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. repetitions are not allowed.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we allow repetition?

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Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x + y)| < 1$,
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Can we do this with 2 colours???

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Monochromatic sumsets in $\mathbb{R}$

Started in **[Hindman, Leader, Strauss 2015]**

If $f : \mathbb{R} \to r$ is **Baire/Lebesgue measurable** then there is a **perfect** $\emptyset \neq X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

Without definability?

There is an $f : \mathbb{R} \to 2$ so that

$$f''X \oplus X = 2 \text{ for every uncountable } X \subset \mathbb{R}.$$

- **[HLS]** using CH, **[Komjáth, DTS, Weiss]** in ZFC, and consistently the number of colours is best possible.

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Continued by [Fernandez-Breton, Rinot 2016]:

There is a colouring $f : \mathbb{R} \to \omega$ so that

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Monochromatic sumsets - with repetitions

Recall: $\exists\, f : \mathbb{N} \to 4$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset \mathbb{N}$.

Let $G(\kappa) = \bigoplus_{\kappa} \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

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Monochromatic sumsets - with repetitions

If $f : G(\kappa) \to r$ then $f \restriction X + X$ is constant for some infinite $X \subset G(\kappa)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.
Notation:

$$G(\kappa) \xrightarrow{+} (\aleph_0)_r$$

Monochromatic sumsets - with repetitions

$\exists\ f : G(\kappa) \to r$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(\kappa)$.

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Notation:

$$G(\kappa) \overset{+}{\not\to} (\aleph_0)_r \text{ e.g. } \mathbb{N} \overset{+}{\not\to} (\aleph_0)_4$$

Monochromatic sumsets - with repetitions

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[Hindman, Leader, Strauss]

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Monochromatic sumsets - with repetitions

$\exists\ f : \mathbb{Q} \to 72$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset \mathbb{Q}$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.

$\exists\ f : G(m) \to 72$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(m)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
- $G(m) \overset{+}{\not\to} (\aleph_0)_{72}$ **for** $m < \omega$.

Monochromatic sumsets - with repetitions

$\exists\ f : G(\aleph_0) \to 144$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(\aleph_0)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
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- $G(\aleph_0) \overset{+}{\not\to} (\aleph_0)_{144}$
-

Monochromatic sumsets - with repetitions

$\exists\ f : G(\aleph_m) \to 2^m \cdot 144$ so that $f \restriction X + X$ is **not constant** for an infinite X.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

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- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
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Corollary

If $2^{\aleph_0} < \aleph_\omega$ then

$$\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$$

for some $r < \omega$.

Positive relations through 'position invariance'

Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

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If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.

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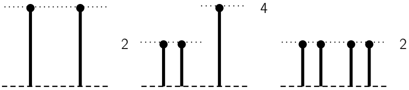

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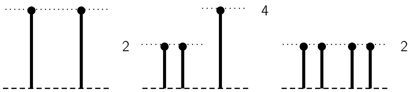

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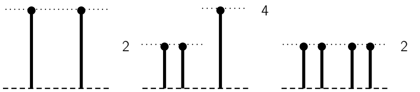

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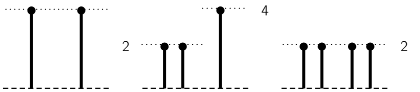

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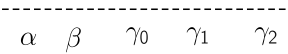

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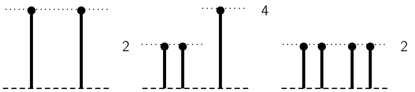

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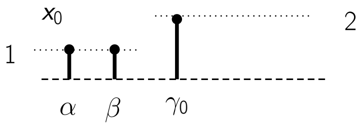

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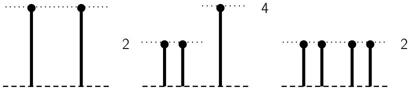

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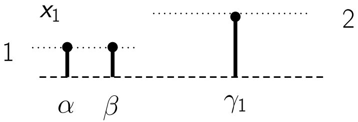

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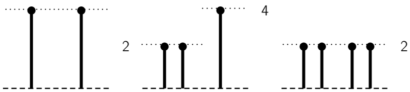

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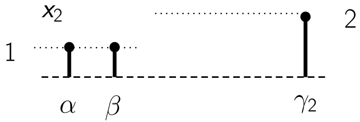

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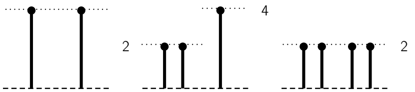

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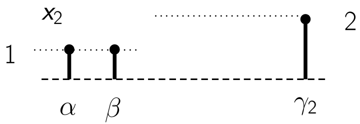

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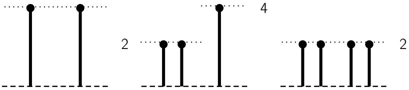

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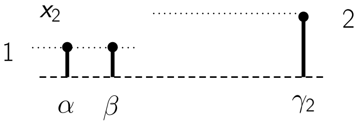

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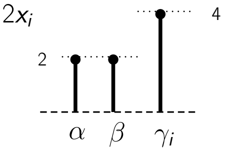

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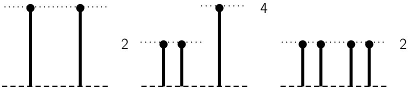

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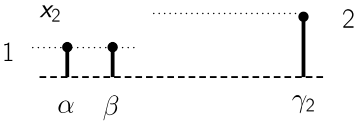

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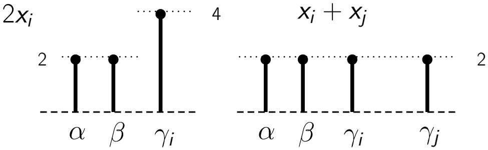

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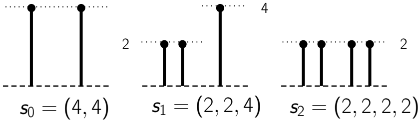

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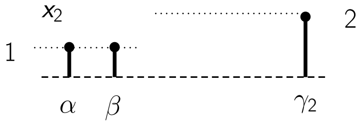

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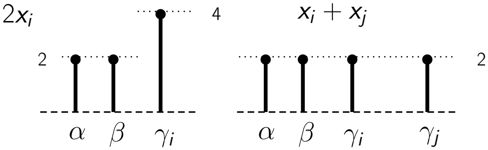

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[Komjáth] and [Leader, Russell] independently

$\Rightarrow G(\kappa) \xrightarrow{+} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

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In ZFC maybe???

- using the Erdős-Rado theorem.

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Consistently, modulo an ω_1-Erdős cardinal,

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Consistently, modulo an ω_1-Erdős cardinal,

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- **[S. Shelah, 2017]**

  "...you can suppose the coloring is continuous, right?"

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- **[S. Shelah, 1988]** Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the type of the finite tree** $F[\bar{x}]$.

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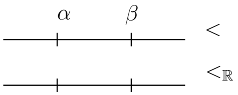

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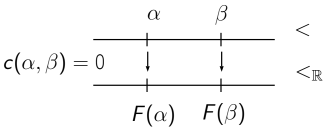

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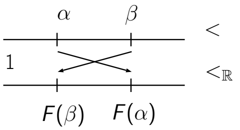

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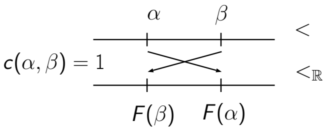

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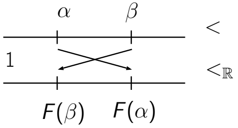

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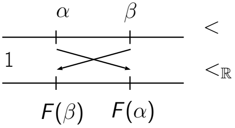

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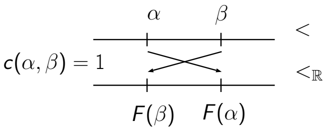

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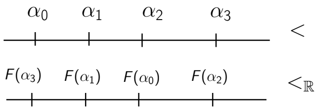

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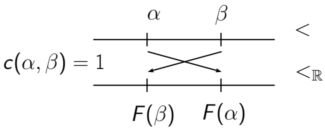

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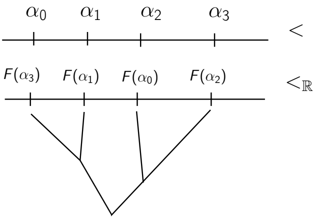

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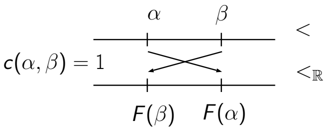

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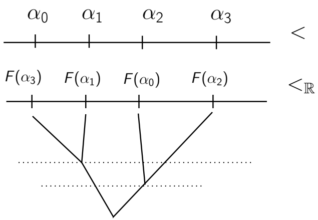

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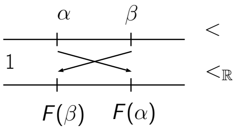

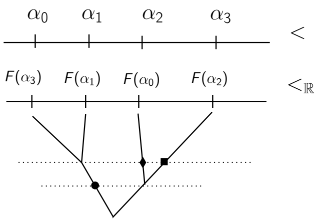

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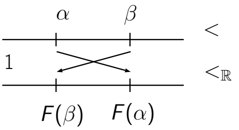

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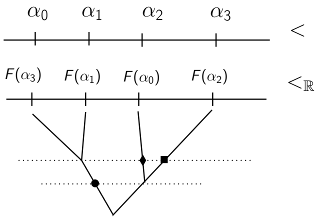

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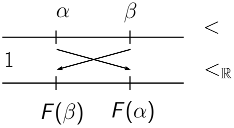

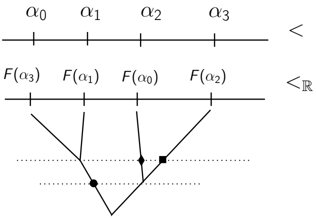

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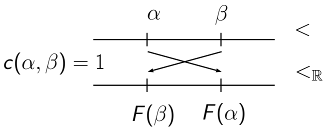

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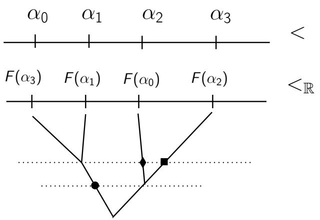

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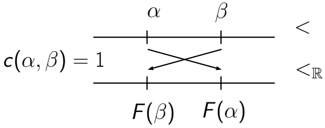

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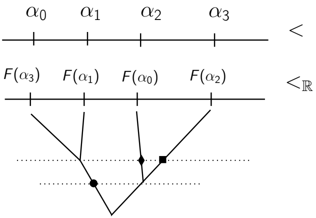

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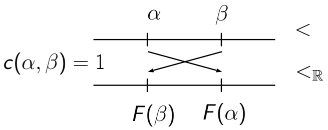

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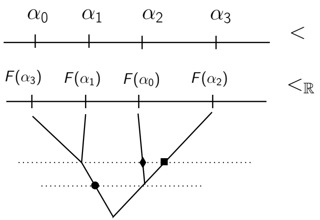

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Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

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Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

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Polarized partition relation in this model: $MA_{\aleph_1}(\text{Knaster}) \Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

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Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

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Finally, look at 4-tuples

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Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

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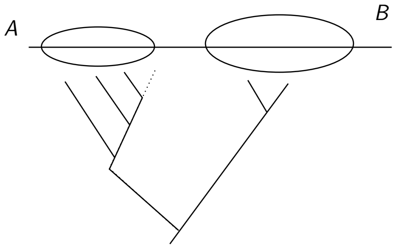

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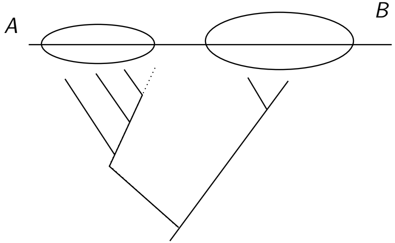

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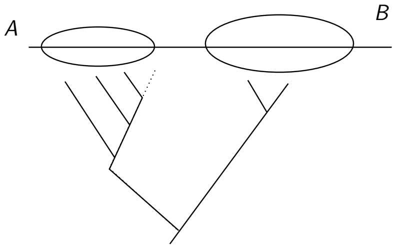

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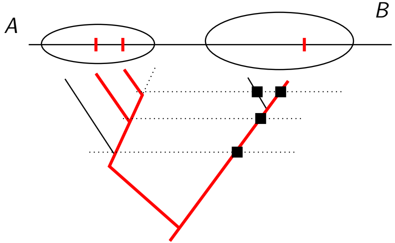

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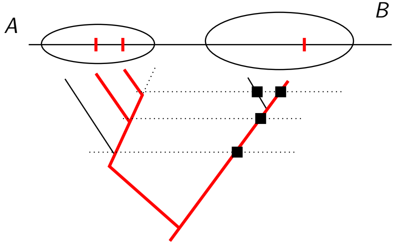

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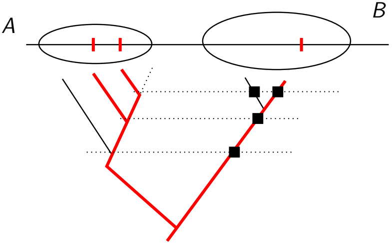

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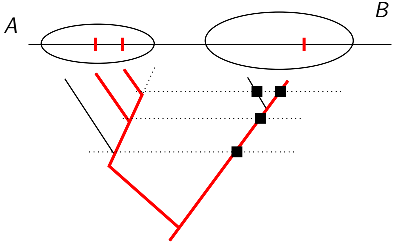

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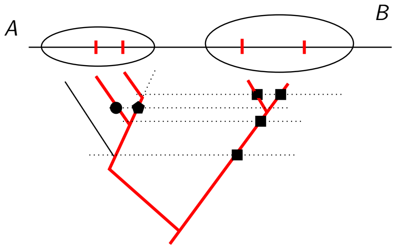

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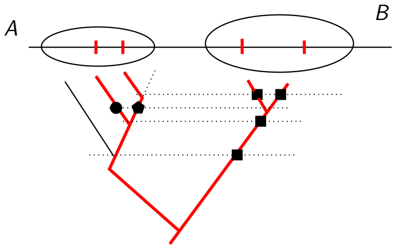

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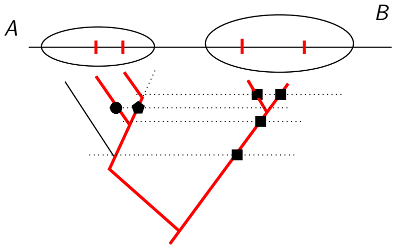

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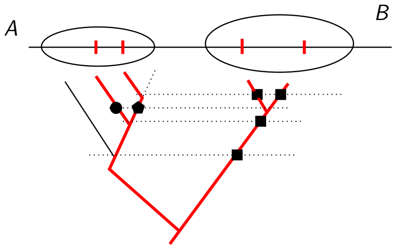

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[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
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- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

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We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
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Various open problems

[Owings, 1974]

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