## Colouring large groups and monochromatic sumsets

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## Ramsey theory: the study of unavoidable regularity

Positive partition relations: a large object, coloured with a small number of colours, always admits monochromatic subsets/substructures of relatively large size.

Szemerédi 1975: any set $A \subset \mathbb{N}$ of positive upper density contains arbitrary long arithmetic progressions.

Shelah 1995: consistently, any function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous on a nowhere meager set.

Negative partition relations: the existence of a colouring without large monochromatic substructures; paradoxical decompositions.

Brown, 1977: there is a function $\mathbb{R} \rightarrow \mathbb{R}$ that is discontinuous on any nowhere measure 0 set.

Komjáth 1994: $\mathbb{R}^{n}$ can be coloured with $\aleph_{0}$ colours so that no two points
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## Consistently, modulo some large cardinal,

if $f: \mathbb{R} \rightarrow r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{R}$ so that
$f \cdot x+X$ is constant.

## P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, Z. Vidnyánszky 2017

## $X+X=\{x+y: x, y \in X\}$ i.e. repetitions are allowed.

- How does this fit into the theory (of partition relations)?
- Why allow repetitions and why only infinite?
- What goes into the proof of this result?

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## Evolving partition relations

... an incomplete overview ...

## Evolving partition relations

If $f: \omega \rightarrow r$ then there is an infinite $X \subset \omega$ with $f \upharpoonright X$ constant.
P. H. Principle

$$
\omega \rightarrow(\omega)_{r}^{1}
$$

## Evolving partition relations

If $f:[\omega]^{k} \rightarrow r$ then there is an infinite $X \subset \omega$ with $f \upharpoonright[X]^{k}$ constant.
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## Evolving partition relations

There is $f:\left[2^{\aleph_{0}}\right]^{2} \rightarrow 2$ so that $f^{\prime \prime}[X]^{2}=2$ for any uncountable $X \subset 2^{\aleph_{0}}$.
P. H. Principle

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## Evolving partition relations

## If $f:\left[\beth_{k-1}^{+}\right]^{k} \rightarrow r$ then $f \upharpoonright[W]^{k}$ is constant for some uncountable $W \subseteq \beth_{k-1}^{+}$.



Erdős, Rado 1956

$$
\beth_{k-1}^{+} \rightarrow\left(\omega_{1}\right)_{r}^{k} \text { for all } r<\omega
$$

## Evolving partition relations

## $F S(X)=\left\{x_{0}+x_{1}+\cdots+x_{\ell}: x_{0}<\cdots<x_{\ell} \in X\right\}$ i.e. no repetitions.



## Evolving partition relations

There is $f:\left[\aleph_{1}\right]^{2} \rightarrow \aleph_{1}$ so that $f^{\prime \prime}[X]^{2}=\aleph_{1}$ for any uncountable $X \subset \aleph_{1}$.
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F. P. Ramsey, 1930

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W. Sierpinski, 1933

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S. Todorcevic, 1987

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## Evolving partition relations

If $f:\left[2^{\aleph_{0}}\right]^{2} \rightarrow 3$ then there is an uncountable $X \subset 2^{\aleph_{0}}$ with $\left|f^{\prime \prime}[X]^{2}\right| \leq 2$.
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## Monochromatic sumsets in $\mathbb{N}$

## Easy Ramsey consequence: if $f: \mathbb{N} \rightarrow r$ with $r \in \omega$ then there is an infinite $X \subset \mathbb{N}$ so that

$$
f \upharpoonright X \oplus X \text { is constant. }
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## Here $X \oplus X=\{x+y: x \neq y \in X\}$ i.e. repetitions are not allowed.

## [Owings, Hindman 1970s] What happens if we allow repetition?

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- if $f: \mathbb{N} \rightarrow r$ then let $g:[\mathbb{N}]^{2} \rightarrow r$ defined by $g(x, y)=f(x+y)$,
- if $X \subset \mathbb{N}$ and $g \upharpoonright[X]^{2}$ is constant then $f \upharpoonright X \oplus X$ is constant too.
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## Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X+X=X \oplus X \cup\{2 x: x \in X\}$.

## There is $f: \mathbb{N} \rightarrow 4$ without infinite monochromatic sumsets:



- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y \ll x \in X$.
- $\left|\log _{\sqrt{2}}(x)-\log _{\sqrt{2}}(x+y)\right|<1$.
- $|f(x)-f(x+y)| \leq 1 \bmod 4$.
- $f(2 x)=\left\lfloor\log _{\sqrt{2}}(x)+2\right\rfloor=f(x)+2 \bmod 4$ so $f(2 x) \neq f(x+y)$.
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Can we do this with 2 colours???

## Monochromatic sumsets in $\mathbb{R}$

## Started in [Hindman, Leader, Strauss 2015] <br> If $f: \mathbb{R} \rightarrow r$ is Baire/Lebesgue measurable then there is a perfect $\emptyset \neq X \subseteq \mathbb{R}$ so that

## $f \upharpoonright X+X$ is constant.

## Without definability?

There is an $f: \mathbb{R} \rightarrow 2$ so that

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f^{\prime \prime} X \oplus X=2 \text { for every uncountable } X \subset \mathbb{R} .
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- [HLS] using CH, [Komjáth, DTS, Weiss] in ZFC, and consistently the number of colours is best possible.


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## Continued by [Fernandez-Breton, Rinot 2016]:

There is a colouring $f: \mathbb{R} \rightarrow \omega$ so that
$f^{\prime \prime} \times \odot X=\omega$ for every $X \subset \mathbb{R}$ of size $c$

For any uncountable, commutative, cancellative semigroup $G$ there is a colouring $f: G \rightarrow \omega$ so that

## Bottom line: without definabilty, infinite sumsets are best possible on $\mathbb{R}$

 with repetition allowed.
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Bottom line: without definabilty, infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

## Any sign of positive relations? Modulo some large cardinals, consistently

## For any $f: \mathbb{R} \rightarrow \omega_{1}$ there is an uncountable subgroup $H \leq \mathbb{R}$ so that

## f $\mathrm{H} H$ has at most ${ }^{\mathrm{N}} 0$ colours.

For any $f: \mathbb{R} \rightarrow 3$ there is an uncountable $X \subset \mathbb{R}$ so that
$f \times x-x$ has at most 2 colours.

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Any sign of positive relations? Modulo some large cardinals, consistently

For any $f: \mathbb{R} \rightarrow \omega_{1}$ there is an uncountable subgroup $H \leq \mathbb{R}$ so that $f \upharpoonright H$ has at most $\aleph_{0}$ colours.

Any sign of positive relations? Modulo some large cardinals, consistently

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For any $f: \mathbb{R} \rightarrow 3$ there is an uncountable $X \subset \mathbb{R}$ so that $f \upharpoonright X \oplus X$ has at most 2 colours.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$. Given $s \in \mathbb{Q}^{<\omega}$ and $a \in[k]^{|s|}$, let
by $\operatorname{supp}(x)=a$ and $x(a(i))=s(i)$.

If $\operatorname{supp}(x)=a$ and $\operatorname{supp}(y)=b$ then


Suppose that $c: \bigoplus_{\kappa} \mathbb{Q} \rightarrow 2$, and let $c_{s}:[\kappa]^{|s|} \rightarrow 2$ by

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- let $\kappa$ be an $\omega_{1}$-Erdős: for any $d:[\kappa]^{<\omega} \rightarrow \theta$, there is an uncountable $X \subset \kappa$ so that $d \upharpoonright[X]^{n}$ is constant for any $n<\omega$.
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- suppose that $\Vdash_{\mathbb{P}} \dot{c}: G(\check{\kappa}) \rightarrow \omega_{1}$, and define $d$ on $[\kappa]^{<\omega}$ by
- this is a countable subset of $\omega_{1}$ by CCC ,
- find uncountable $X \subset \mathcal{K}$ so that $d\left\lceil[X]^{n}\right.$ is constant $I_{n}$ for any $n<\omega$,
- let $H=\{x \in G(\kappa): \operatorname{supp}(x) \subset X\}$, and now

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- let $H=\{x \in G(\kappa): \operatorname{supp}(x) \subset X\}$, and now

$$
\Vdash_{\mathbb{P}} r \operatorname{ran}(\dot{c} \upharpoonright H) \subseteq \bigcup_{n \in \omega} I_{n} .
$$

## Monochromatic sumsets - with repetitions

Recall: $\exists f: \mathbb{N} \rightarrow 4$ so that $f \mid X+X$ is not constant for an infinite $X \subset \mathbb{N}$.

## Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.

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## Monochromatic sumsets - with repetitions

If $f: G(\kappa) \rightarrow r$ then $f \upharpoonright X+X$ is constant for some infinite $X \subset G(\kappa)$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$. Notation:

$$
G(\kappa) \xrightarrow{+}\left(\aleph_{0}\right)_{r}
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## Monochromatic sumsets - with repetitions

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[Hindman, Leader, Strauss]

## Monochromatic sumsets - with repetitions

$\exists f: \mathbb{Q} \rightarrow 72$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset \mathbb{Q}$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
[Hindman, Leader, Strauss]

- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.


## Monochromatic sumsets - with repetitions

$\exists f: G(m) \rightarrow 72$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset G(m)$.

$$
\text { Let } G(\kappa)=\bigoplus_{\kappa} \mathbb{Q} \text { i.e. } x: \kappa \rightarrow \mathbb{Q} \text { with }|\operatorname{supp}(x)|<\omega \text {. E.g. } G\left(2^{\aleph_{0}}\right) \approx \mathbb{R} \text {. }
$$

[Hindman, Leader, Strauss]

- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.
- $G(m) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$ for $m<\omega$.


## Monochromatic sumsets - with repetitions

$\exists f: G\left(\aleph_{0}\right) \rightarrow 144$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset G\left(\aleph_{0}\right)$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
[Hindman, Leader, Strauss]

- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.
- $G(m) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$ for $m<\omega$.
- $G\left(\aleph_{0}\right) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{144}$


## Monochromatic sumsets - with repetitions

$\exists f: G\left(\aleph_{m}\right) \rightarrow 2^{m} \cdot 144$ so that $f \upharpoonright X+X$ is not constant for an infinite $X$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
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## Monochromatic sumsets - with repetitions

$\exists f: \mathbb{R} \rightarrow r$ so that $f \upharpoonright X+X$ is not constant for an infinite $X$.

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## Corollary

If $2^{\aleph_{0}}<\aleph_{\omega}$ then

$$
\mathbb{R} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{r}
$$

for some $r<\omega$.

```
Given }s\in\mp@subsup{\mathbb{Q}}{}{<\omega}\mathrm{ and }a\in[\kappa\mp@subsup{]}{}{|s|}\mathrm{ , let
by }\operatorname{supp}(x)=a\mathrm{ and }x(a(i))=s(i)
Suppose that c: (D) O) }->2\mathrm{ , and let
cs:[k] [s]}->2\mathrm{ by
cs(a) =c(s*a).
by \(\operatorname{supp}(x)=a\) and \(x(a(i))=s(i)\).
Suppose that \(c:(0) \rightarrow 2\) and let \(c_{5}:[k]^{|s|} \rightarrow 2\) by \(c_{s}(a)=c(s * a)\).
```

```
If }\kappa\mathrm{ is large enough then there is a
```

If }\kappa\mathrm{ is large enough then there is a
large W\subseteq\kappa so that }\mp@subsup{c}{\mp@subsup{s}{i}{}}{}\mathrm{ are constant
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on W for i=0,1,2.

```
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Assume that $c_{s_{1}}$ and $c_{s_{2}}$ are both constant 0 .

## Positive relations through 'position invariance'

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Given }s\in\mp@subsup{\mathbb{Q}}{}{<\omega}\mathrm{ and }a\in[\kappa\mp@subsup{]}{}{|s|}\mathrm{ , let
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```
Suppose that \(c: \oplus_{\kappa} \mathbb{Q} \rightarrow 2\), and let \(c_{s}:[k]^{s \mid} \rightarrow 2\) by
If \(\kappa\) is large enough then there is a large \(W \subseteq \kappa\) so that \(c_{s_{i}}\) are constant on \(1 /\) for \(i=0,1,2\)
```


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Let $a_{i}=\left\{\alpha, \beta, \gamma_{i}\right\}$ and $x_{i}=\frac{1}{2} s_{1} * a_{i}$.

2


## Positive relations through 'position invariance'

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If $c_{s_{0}}, c_{s_{2}}$ have the same constant then we need $\operatorname{tp}(W)=\omega+\omega$.

## Corollaries

## [Komjáth] and [Leader, Russell] independently

$\Rightarrow G(\kappa) \xrightarrow{+}\left(\aleph_{0}\right)_{r}$ where $\kappa=\beth_{2 r-1}\left(\aleph_{0}\right)$,


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## Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_{0}}<\aleph_{\omega}$ then $\mathbb{R} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{r}$ for some $r<\omega$.

## Consistently, modulo an $\omega_{1}$-Erdős cardinal,

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- [S. Shelah, 2017]
"...you can suppose the coloring is continuous, right?"


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- [S. Shelah, 1988] Consistently, modulo an $\omega_{1}$-Erdős cardinal, if $f:\left[2^{\aleph_{0}}\right]^{<\omega} \rightarrow r$ then there is an uncountable $X$ and $F: X \hookrightarrow 2^{\omega}$ so that $f(\bar{x})$ only depends on the type of the finite tree $F[\bar{x}]$.


## Positive partition relations on $\kappa=2^{\aleph_{0}}$ ? No way...

```
Sierpinski colouring: c: [2 ^
so that c(\alpha,\beta)=0 iff
\alpha<\beta\leftrightarrowF(\alpha)<\mp@subsup{\mathbb{R}}{}{\prime}F(\beta)
for some fixed F: 2 }\mp@subsup{}{}{\mp@subsup{\aleph}{0}{}}\hookrightarrow\mathbb{R}\simeq\mp@subsup{2}{}{\omega
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Also: $\Delta^{\prime \prime}[X]^{2}$ is infinite for any infinite $X \subseteq 2^{\omega}$. Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^{i}$ iff for all $I_{1}, I_{2}, I_{3}, I_{4}<i$ :

```
2 colours on any uncountable set!
```

You can define more complicated

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        c:[2\mp@subsup{N}{0}{}}\mp@subsup{]}{}{k}->
using an F}:\mp@subsup{2}{}{\mp@subsup{N}{0}{}}\hookrightarrow\mathbb{R}\simeq\mp@subsup{2}{}{\omega}\mathrm{ and the
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\Delta(x, y)=\min \{n: x(n) \neq y(n)\} .
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## Positive partition relations on $\kappa=2^{\aleph_{0}}$ ? No way...

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Also: $\Delta^{\prime \prime}[X]^{2}$ is infinite for any infinite $X \subseteq 2^{\omega}$.

2 colours on any uncountable set!
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- $\Delta\left(\bar{t}\left(I_{1}\right), \bar{t}\left(I_{2}\right)\right)<\Delta\left(\bar{t}\left(I_{3}\right), \bar{t}\left(I_{4}\right)\right)$ iff $\Delta\left(\bar{s}\left(I_{1}\right), \bar{s}\left(I_{2}\right)\right)<\Delta\left(\bar{s}\left(I_{3}\right), \bar{s}\left(I_{4}\right)\right)$,
- $\bar{z}\left(I_{3}\right) \upharpoonright n<\operatorname{lex} \bar{t}\left(I_{4}\right) \upharpoonright n$ for $n=\Delta\left(\bar{t}\left(I_{1}\right), \bar{t}\left(I_{2}\right)\right)$
- $\bar{t}\left(I_{3}\right)(n)=0$ for $n=\Delta\left(\bar{t}\left(I_{1}\right), \bar{t}\left(I_{2}\right)\right)$

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$\bar{s}\left(I_{3}\right) \upharpoonright m<_{\text {lex }} \bar{s}\left(I_{4}\right) \upharpoonright m$ for $m=\Delta\left(\bar{s}\left(I_{1}\right), \bar{s}\left(I_{2}\right)\right)$,
- $\bar{t}\left(l_{3}\right)(n)=0$ for $n=\Delta\left(\bar{t}\left(I_{1}\right), \bar{t}\left(I_{2}\right)\right)$
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## Shelah's 'Was Sierpinski right?' papers

```
We cannot realize more colours:
[Shclah, M/SR 1]
Consistently, modulo an }\mp@subsup{\omega}{1}{}\mathrm{ -Erdós cardinal, for
any c:[ [2No}\mp@subsup{]}{}{2}->r\mathrm{ there is an uncountable
W \subset 2 ^ { N 0 } \text { with at most } 2 \text { colours.}
Larger tuples can define more colours... What
was so specific about the colourings before?
if c:[2N0}\mp@subsup{]}{}{k}->r\mathrm{ and }F:\mp@subsup{2}{}{\mp@subsup{N}{0}{}}\hookrightarrow\mathbb{R}\simeq\mp@subsup{2}{}{\omega
c(\overline{\alpha})=c(\overline{\beta})\mathrm{ whenever F(晾) }~F(\overline{\beta}).
```


## [Shelah, WSR II]

Suppose that $\lambda$ is an $\omega_{1}$-Erdós cardinal in $V$
Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$
satisfies the following:

- $2^{\aleph_{0}}=\lambda$
- $\mathrm{MA}_{\aleph_{1}}$ (Knaster), and
then there is $W \in[\lambda]^{\aleph_{1}}$ and $F: W \hookrightarrow \mathbb{R} \simeq 2^{W}$
so that

```
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then c}\mathrm{ is F-canonical on W}\subseteq\mp@subsup{2}{}{\mp@subsup{\aleph}{0}{}}\mathrm{ iff
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Polarized partition relation in this model: \(\mathrm{MA}_{\aleph_{1}}\) (Knaster) \(\Rightarrow\) if \(g:[\omega]^{2} \times \omega_{1} \rightarrow 2\) then there
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\section*{Shelah's 'Was Sierpinski right?' papers}

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Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, for any \(c:\left[2^{\aleph_{0}}\right]^{2} \rightarrow r\) there is an uncountable \(W \subset 2^{\text {No }}\) with at most 2 colours.

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if \(c:\left[2^{\aleph_{0}}\right]^{k} \rightarrow r\) and \(F: 2^{N_{0}} \hookrightarrow \mathbb{R} \simeq 2^{\omega}\)
\(c_{i}\) is \(F\)-canonical on \(W\).
\(c(\bar{\alpha})=c(\bar{\beta})\) whenever \(F(\bar{\alpha}) \sim F(\bar{\beta})\).

Polarized partition relation in this model: \(\mathrm{MA}_{\aleph_{1}}\) (Knaster) \(\Rightarrow\) if \(g:[\omega]^{2} \times \omega_{1} \rightarrow 2\) then there is \(A \in[\omega]^{\omega}, B \in\left[\omega_{1}\right]^{\omega_{1}}\) so that \(g \upharpoonright[A]^{2} \times B\) is constant.

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\(\left.c^{\prime} \bar{a}\right)=c^{\prime}(\bar{\rho})\) whenever \(\Gamma^{\prime}(\bar{a}) \sim \Gamma^{\prime}(\bar{\rho})\)

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if \(c:\left[2^{\aleph_{0}}\right]^{k} \rightarrow r\) and \(F: 2^{\aleph_{0}} \hookrightarrow \mathbb{R} \simeq 2^{\omega}\)
then \(c\) is F-canonical on \(M \subset 2^{K_{0}}\) iff
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\[
\begin{aligned}
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if \(c:\left[2^{\aleph_{0}}\right]^{k} \rightarrow r\) and \(F: 2^{N_{0}} \hookrightarrow \mathbb{R} \simeq 2^{\omega}\) then \(c\) is \(F\)-canonical on \(W \subseteq 2^{\aleph_{0}}\) iff \(c(\bar{\alpha})=c(\bar{\beta})\) whenever \(F(\bar{\alpha}) \sim F(\bar{\beta})\).
[Shelah, WSR II]
Suppose that \(\lambda\) is an \(\omega_{1}\)-Erdős cardinal in \(V\).
Then there is a forcing notion \(\mathbb{P}\) so that \(V^{\mathbb{P}}\) satisfies the following:
- \(2^{N_{0}}=\lambda\),
- \(\mathrm{MA}_{\aleph_{1}}\) (Knaster), and
if \(c_{i}:[\lambda]^{i} \rightarrow r\) for \(i<k<\omega, r<\omega\), then there is \(W \in[\lambda]^{\aleph_{1}}\) and \(F: W \hookrightarrow \mathbb{R} \simeq 2^{\omega}\) so that
\[
c_{i} \text { is } F \text {-canonical on } W \text {. }
\]

\section*{Shelah's 'Was Sierpinski right?' papers}

We cannot realize more colours:

\section*{[Shelah, WSR I]}

Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, for any \(c:\left[2^{\aleph 0}\right]^{2} \rightarrow r\) there is an uncountable \(W \subseteq 2^{\aleph_{0}}\) with at most 2 colours.

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Polarized partition relation in this model:
is \(A \in[\omega]^{\omega}, B \in\left[\omega_{1}\right]^{\omega_{1}}\) so that \(g \upharpoonright[A]^{2} \times B\) is constant.

\section*{Shelah's 'Was Sierpinski right?' papers}

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Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, for any \(c:\left[2^{\aleph 0}\right]^{2} \rightarrow r\) there is an uncountable \(W \subseteq 2^{\aleph_{0}}\) with at most 2 colours.

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\(c_{i}\) is \(F\)-canonical on \(W\).

Polarized partition relation in this model: \(\mathrm{MA}_{\aleph_{1}}\) (Knaster) \(\Rightarrow\) if \(g:[\omega]^{2} \times \omega_{1} \rightarrow 2\) then there is \(A \in[\omega]^{\omega}, B \in\left[\omega_{1}\right]^{\omega_{1}}\) so that \(g \upharpoonright[A]^{2} \times B\) is constant.

\section*{Proving \(G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}\) in the 'WSR II' model}

Take \(c: G\left(2^{\aleph_{0}}\right) \rightarrow 2\) and consider \(c_{s_{0}}, c_{s_{1}}, c_{s_{2}}\)
with \(s_{0}=(4,4), s_{1}=(2,2,4), s_{2}=(2,2,2,2)\).
Apply M/SR II. there is \(|M /|=\aleph_{1}\) and \(F: W \hookrightarrow \mathbb{R}\) so that \(c_{s_{i}}\) is \(F\)-canonical on \(W\). Select \(|A|=\aleph_{0},|B|=\aleph_{1}\) from \(W\) so that \(A<B\) and \(F^{\prime \prime} A<\mathbb{R} F^{\prime \prime} B\).
```

How can we fix the type of triples

```
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(\alpha,\mp@subsup{\alpha}{}{\prime},\beta)\in\mp@subsup{A}{}{2}\timesB?
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Let g: A
Let g: A
g(\alpha,\mp@subsup{\alpha}{}{\prime},\beta)=F(\beta)(m)
g(\alpha,\mp@subsup{\alpha}{}{\prime},\beta)=F(\beta)(m)
with m=\Delta(F(\alpha),F(\mp@subsup{\alpha}{}{\prime})). Shrink using the
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polarized relation to fix the type!
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=>C}\mp@subsup{C}{\mp@subsup{S}{1}{}}{}\mathrm{ is constant too on these triples.
=>C}\mp@subsup{C}{\mp@subsup{S}{1}{}}{}\mathrm{ is constant too on these triples.
Finally, look at 4-tuples
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    (\alpha,\mp@subsup{\alpha}{}{\prime},\beta,\mp@subsup{\beta}{}{\prime})\in\mp@subsup{A}{}{2}\times\mp@subsup{B}{}{2}.
    (\alpha,\mp@subsup{\alpha}{}{\prime},\beta,\mp@subsup{\beta}{}{\prime})\in\mp@subsup{A}{}{2}\times\mp@subsup{B}{}{2}.
Look at splitting levels from B, read values on
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branches from }A\mathrm{ , thin both to fix the values.
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This fixes the type of these 4-tuples too on
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some countable A, B.
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C}\mp@subsup{c}{\mp@subsup{s}{2}{}}{}\mathrm{ is constant on these 4-tuples.
C}\mp@subsup{c}{\mp@subsup{s}{2}{}}{}\mathrm{ is constant on these 4-tuples.
Now, two constant values must agree of the three; repeat the first trick to construct infinite \(X\) so that \(X+X\) is monochromatic.
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```

$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same
$\sim$-type, so $C_{S_{n}}$ is constant.


## Proving $G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}$ in the 'WSR II' model

Take $c: G\left(2^{\aleph_{0}}\right) \rightarrow 2$ and consider $c_{s_{0}}, c_{s_{1}}, c_{s_{2}}$ with $s_{0}=(4,4), s_{1}=(2,2,4), s_{2}=(2,2,2,2)$.

Apply WSR II: there is $|W|=\aleph_{1}$ and
$F: W \hookrightarrow \mathbb{R}$ so that $c_{s_{i}}$ is $F$-canonical on $W$

## Select $|A|=\aleph_{0},|B|=\aleph_{1}$ from $W$ so that

$A<B$ and $F^{\prime \prime} A<\infty F^{\prime \prime} B$.

```
How can we fix the type of triples
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Let $g: A^{2} \times B \rightarrow 2$ by

with $m=\Delta\left(F(\alpha), F\left(\alpha^{\prime}\right)\right)$. Shrink using thepolarized relation to fix the type!
$\Rightarrow C_{S_{7}}$ is constant too on these triples.
Finally, look at 4-tuples
$\left(\alpha, \alpha^{\prime}, \beta, \beta^{\prime}\right) \in A^{2} \times B^{2}$

Look at splitting levels from $B$, read values on branches from $A$, thin both to fix the values. This fixes the type of these 4-tuples too on some countable $A, B$.
$\Rightarrow c_{s_{2}}$ is constant on these 4-tuples.
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## Proving $G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}$ in the 'WSR II' model

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How can we fix the type of triples

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How can we fix the type of triples


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$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so $c_{s_{0}}$ is constant.

How can we fix the type of triples

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## Proving $G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}$ in the 'WSR II' model

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## Proving $G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}$ in the 'WSR II' model

Take $c: G\left(2^{\aleph_{0}}\right) \rightarrow 2$ and consider $c_{s_{0}}, c_{s_{1}}, c_{s_{2}}$ with $s_{0}=(4,4), s_{1}=(2,2,4), s_{2}=(2,2,2,2)$.
Apply WSR II: there is $|W|=\aleph_{1}$ and $F: W \hookrightarrow \mathbb{R}$ so that $c_{s_{i}}$ is $F$-canonical on $W$.

Select $|A|=\aleph_{0},|B|=\aleph_{1}$ from $W$ so that $A<B$ and $F^{\prime \prime} A<\mathbb{R} F^{\prime \prime} B$.

$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so $c_{s_{0}}$ is constant.

How can we fix the type of triples

$$
\left(\alpha, \alpha^{\prime}, \beta\right) \in A^{2} \times B ?
$$

Let $g: A^{2} \times B \rightarrow 2$ by

$$
g\left(\alpha, \alpha^{\prime}, \beta\right)=F(\beta)(m)
$$

with $m=\Delta\left(F(\alpha), F\left(\alpha^{\prime}\right)\right)$. Shrink using the polarized relation to fix the type! $\Rightarrow c_{s_{1}}$ is constant too on these triples.

Finally, look at 4-tuples

$$
\left(\alpha, \alpha^{\prime}, \beta, \beta^{\prime}\right) \in A^{2} \times B^{2} .
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Look at splitting levels from $B$, read values on branches from $A$, thin both to fix the values. This fixes the type of these 4-tuples too on some countable $A, B$.
$\Rightarrow c_{s_{2}}$ is constant on these 4-tuples.
Now, two constant values must agree of the
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## Various open problems

## [Owings, 1974]

- $\mathbb{N} / \mathrm{A}\left(\mathrm{N}_{0}\right)_{2}$ ???

Connected to our results:

## Monochromatic $k$-sumsets: $X+X+\cdots+X$ ?

[HLS] There is a finite colouring of $G\left(\aleph_{n}\right)$ with no infinite monochromatic $k$-sumsets $(n<\omega)$,
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We are far from a complete picture.
[Shelah, 1988]

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[^0]:    2 colours on any uncountable set!

