Monochromatic sumsets for colourings of $\mathbb{R}$

Dániel T. Soukup

http://www.logic.univie.ac.at/∼soukupd73/

Supported in part by **FWF Grant I1921**.

Introduction

Consistently, modulo some large cardinal,

if $f : \mathbb{R} \to r$ with $r \in \omega$ then there is an **infinite** $X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

$X + X = \{x + y : x, y \in X\}$ i.e. repetitions are allowed.

- How does this fit into the theory of partition relations?
- What goes into the proof of this result?
- Joint result with P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, and Z. Vidnyánszky.

Introduction

Consistently, modulo some large cardinal,

if $f : \mathbb{R} \to r$ with $r \in \omega$ then there is an **infinite** $X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

$X + X = \{x + y : x, y \in X\}$ i.e. **repetitions are allowed**.

- How does this fit into the theory of partition relations?
- What goes into the proof of this result?
- Joint result with P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, and Z. Vidnyánszky.

Introduction

Consistently, modulo some large cardinal,

if $f : \mathbb{R} \to r$ with $r \in \omega$ then there is an **infinite** $X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \textbf{ is constant.}$$

$X + X = \{x + y : x, y \in X\}$ i.e. **repetitions are allowed**.

- How does this fit into the theory of partition relations?
- What goes into the proof of this result?
- Joint result with P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, and Z. Vidnyánszky.

Introduction

Consistently, modulo some large cardinal,

if $f : \mathbb{R} \to r$ with $r \in \omega$ then there is an **infinite** $X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

$X + X = \{x + y : x, y \in X\}$ i.e. **repetitions are allowed**.

- How does this fit into the theory of partition relations?
- What goes into the proof of this result?
- Joint result with P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, and Z. Vidnyánszky.

Introduction

Consistently, modulo some large cardinal,

if $f : \mathbb{R} \to r$ with $r \in \omega$ then there is an **infinite** $X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

$X + X = \{x + y : x, y \in X\}$ i.e. **repetitions are allowed**.

- How does this fit into the theory of partition relations?
- What goes into the proof of this result?
- Joint result with P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T. Soukup, and Z. Vidnyánszky.

Evolving partition relations

... an incomplete overview ...

Evolving partition relations

If $f : \omega \to r$ then there is an infinite $X \subset \omega$ with $f \restriction X$ constant.

P. H. Principle

$$\omega \to (\omega)^1_r$$

Evolving partition relations

If $f : [\omega]^k \to r$ then there is an infinite $X \subset \omega$ with $f \restriction [X]^k$ constant.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

Evolving partition relations

There is $f:[2^{\aleph_0}]^2 \to 2$ so that $f''[X]^2 = 2$ for any uncountable $X \subset 2^{\aleph_0}$.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

W. Sierpinski, 1933

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Evolving partition relations

If $f : [\beth_{k-1}^+]^k \to r$ then $f \restriction [W]^k$ is constant for some uncountable $W \subseteq \beth_{k-1}^+$.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

W. Sierpinski, 1933

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Erdős, Rado 1956

$\beth_{k-1}^+ \to (\omega_1)^k_r$ for all $r < \omega$.

Evolving partition relations

$FinSum(X) = \{x_0 + x_1 + \cdots + x_\ell : x_0 < \cdots < x_\ell \in X\}$ i.e. no repetitions.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

W. Sierpinski, 1933

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Erdős, Rado 1956

$\beth^+_{k-1} \to (\omega_1)^k_r$ for all $r < \omega$.

N. Hindman, 1974

if $f : \mathbb{N} \to r$ then there is some infinite $X \subseteq \mathbb{N}$ so that $f \restriction FinSum(X)$ is constant.

Evolving partition relations

There is $f : [\aleph_1]^2 \to \aleph_1$ so that $f''[X]^2 = \aleph_1$ for any uncountable $X \subset \aleph_1$.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

W. Sierpinski, 1933

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Erdős, Rado 1956

$\beth^+_{k-1} \to (\omega_1)^k_r$ for all $r < \omega$.

N. Hindman, 1974

if $f : \mathbb{N} \to r$ then there is some infinite $X \subseteq \mathbb{N}$ so that $f \restriction FinSum(X)$ is constant.

S. Todorcevic, 1987

$$\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$$

Evolving partition relations

If $f : [2^{\aleph_0}]^2 \to 3$ then there is an uncountable $X \subset 2^{\aleph_0}$ with $|f''[X]^2| \leq 2$.

P. H. Principle

$$\omega \to (\omega)^1_r$$

F. P. Ramsey, 1930

$$\omega \to (\omega)^k_r$$

W. Sierpinski, 1933

$$2^{\aleph_0} \not\to (\aleph_1)^2_2$$

Erdős, Rado 1956

$\beth^+_{k-1} \to (\omega_1)^k_r$ for all $r < \omega$.

N. Hindman, 1974

if $f : \mathbb{N} \to r$ then there is some infinite $X \subseteq \mathbb{N}$ so that $f \restriction FinSum(X)$ is constant.

S. Todorcevic, 1987

$$\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$$

S. Shelah, 1988

$$\mathrm{Con}(2^{\aleph_0} \to [\aleph_1]^2_3)$$

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we allow repetition?

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we **allow repetition**?

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we allow repetition?

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we **allow repetition**?

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we allow repetition?

Monochromatic sumsets in $\mathbb{N}$

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we allow repetition?

Easy Ramsey consequence: if $f : \mathbb{N} \to r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$f \restriction X \oplus X \text{ is constant.}$$

Here $X \oplus X = \{x + y : x \neq y \in X\}$ i.e. **repetitions are not allowed**.

Proof:

- if $f : \mathbb{N} \to r$ then let $g : [\mathbb{N}]^2 \to r$ defined by $g(x, y) = f(x + y)$,
- if $X \subset \mathbb{N}$ and $g \restriction [X]^2$ is constant then $f \restriction X \oplus X$ is constant too.

[Owings, Hindman 1970s] What happens if we allow repetition?

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \bmod 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \bmod 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \bmod 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X + X = X \oplus X \cup \{2x : x \in X\}$.

There is $f : \mathbb{N} \to 4$ without infinite monochromatic sumsets:

$$f(x) = \lfloor \log_{\sqrt{2}}(x) \rfloor \text{ mod } 4.$$

- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y << x \in X$.
- $|\log_{\sqrt{2}}(x) - \log_{\sqrt{2}}(x+y)| < 1$,
- $|f(x) - f(x+y)| \leq 1 \text{ mod } 4$.
- $f(2x) = \lfloor \log_{\sqrt{2}}(x) + 2 \rfloor = f(x) + 2 \text{ mod } 4$ so $f(2x) \neq f(x+y)$.

Can we do this with 2 colours???

Monochromatic sumsets in $\mathbb{R}$

Started in **[Hindman, Leader, Strauss 2015]**

If $f : \mathbb{R} \to r$ is **Baire/Lebesgue measurable** then there is a **perfect** $\emptyset \neq X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

Without definability?

There is an $f : \mathbb{R} \to 2$ so that

$$f''X \oplus X = 2 \text{ for every uncountable } X \subset \mathbb{R}.$$

- **[HLS]** using CH, **[Komjáth, DTS, Weiss]** in ZFC, and consistently the number of colours is best possible.

Monochromatic sumsets in $\mathbb{R}$

Started in **[Hindman, Leader, Strauss 2015]**

If $f : \mathbb{R} \to r$ is **Baire/Lebesgue measurable** then there is a **perfect** $\emptyset \neq X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

Without definability?

There is an $f : \mathbb{R} \to 2$ so that

$$f''X \oplus X = 2 \textbf{ for every uncountable } X \subset \mathbb{R}.$$

- **[HLS]** using CH, **[Komjáth, DTS, Weiss]** in ZFC, and consistently the number of colours is best possible.

Monochromatic sumsets in $\mathbb{R}$

Started in **[Hindman, Leader, Strauss 2015]**

If $f : \mathbb{R} \to r$ is **Baire/Lebesgue measurable** then there is a **perfect** $\emptyset \neq X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

Without definability?

There is an $f : \mathbb{R} \to 2$ so that

$$f''X \oplus X = 2 \text{ for every uncountable } X \subset \mathbb{R}.$$

- **[HLS]** using CH, **[Komjáth, DTS, Weiss]** in ZFC, and consistently the number of colours is best possible.

Monochromatic sumsets in $\mathbb{R}$

Started in **[Hindman, Leader, Strauss 2015]**

If $f : \mathbb{R} \to r$ is **Baire/Lebesgue measurable** then there is a **perfect** $\emptyset \neq X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

Without definability?

There is an $f : \mathbb{R} \to 2$ so that

$$f''X \oplus X = 2 \text{ **for every uncountable** } X \subset \mathbb{R}.$$

- [HLS] using CH, [Komjáth, DTS, Weiss] in ZFC, and consistently the number of colours is best possible.

Monochromatic sumsets in $\mathbb{R}$

Started in **[Hindman, Leader, Strauss 2015]**

If $f : \mathbb{R} \to r$ is **Baire/Lebesgue measurable** then there is a **perfect** $\emptyset \neq X \subseteq \mathbb{R}$ so that

$$f \restriction X + X \text{ is constant.}$$

Without definability?

There is an $f : \mathbb{R} \to 2$ so that

$$f''X \oplus X = 2 \textbf{ for every uncountable } X \subset \mathbb{R}.$$

- **[HLS]** using CH, **[Komjáth, DTS, Weiss]** in ZFC, and consistently the number of colours is best possible.

Monochromatic sumsets in $\mathbb{R}$

Continued by **[Fernandez-Breton, Rinot 2016]**:

- how to realize more colours on sets of the form $FinSum(X)$ (no repetitions),
- general theorems on **uncountable, commutative, cancellative semigroups** G.

Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

Monochromatic sumsets in $\mathbb{R}$

Continued by **[Fernandez-Breton, Rinot 2016]**:

- how to realize more colours on sets of the form $FinSum(X)$ (no repetitions),
- general theorems on **uncountable, commutative, cancellative semigroups** G.

Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

Monochromatic sumsets in $\mathbb{R}$

Continued by **[Fernandez-Breton, Rinot 2016]**:

- how to realize more colours on sets of the form $FinSum(X)$ (no repetitions),
- general theorems on **uncountable, commutative, cancellative semigroups** G.

Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

Monochromatic sumsets in $\mathbb{R}$

Continued by **[Fernandez-Breton, Rinot 2016]**:

- how to realize more colours on sets of the form $FinSum(X)$ (no repetitions),
- general theorems on **uncountable, commutative, cancellative semigroups** G.

Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

Monochromatic sumsets in $\mathbb{R}$

Continued by **[Fernandez-Breton, Rinot 2016]**:

- how to realize more colours on sets of the form $FinSum(X)$ (no repetitions),
- general theorems on **uncountable, commutative, cancellative semigroups** G.

Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

Monochromatic sumsets in $\mathbb{R}$

Continued by **[Fernandez-Breton, Rinot 2016]**:

- how to realize more colours on sets of the form $FinSum(X)$ (no repetitions),
- general theorems on **uncountable, commutative, cancellative semigroups** G.

Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

Monochromatic sumsets - with repetitions

Recall: $\exists f : \mathbb{N} \to 4$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset \mathbb{N}$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

Monochromatic sumsets - with repetitions

Recall: $\exists\ f : \mathbb{N} \to 4$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset \mathbb{N}$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

Monochromatic sumsets - with repetitions

If $f : G(\kappa) \to r$ then $f \restriction X + X$ is constant for some infinite $X \subset G(\kappa)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.
Notation:

$$G(\kappa) \xrightarrow{+} (\aleph_0)_r$$

Monochromatic sumsets - with repetitions

$\exists\ f : G(\kappa) \to r$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(\kappa)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.
Notation:

$$G(\kappa) \overset{+}{\not\to} (\aleph_0)_r \text{ e.g. } \mathbb{N} \overset{+}{\not\to} (\aleph_0)_4$$

Monochromatic sumsets - with repetitions

$\exists\, f : G(\kappa) \to r$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(\kappa)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

Monochromatic sumsets - with repetitions

$\exists\, f : \mathbb{Q} \to 72$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset \mathbb{Q}$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.

Monochromatic sumsets - with repetitions

$\exists\, f : G(m) \to 72$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(m)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
- $G(m) \overset{+}{\not\to} (\aleph_0)_{72}$ **for** $m < \omega$.

$\exists\ f : G(\aleph_0) \to 144$ so that $f \restriction X + X$ is **not constant** for an infinite $X \subset G(\aleph_0)$.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
- $G(m) \overset{+}{\not\to} (\aleph_0)_{72}$ for $m < \omega$.
- $G(\aleph_0) \overset{+}{\not\to} (\aleph_0)_{144}$
-

Monochromatic sumsets - with repetitions

$\exists\ f : G(\aleph_m) \to 2^m \cdot 144$ so that $f \restriction X + X$ is **not constant** for an infinite X.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
- $G(m) \overset{+}{\not\to} (\aleph_0)_{72}$ for $m < \omega$.
- $G(\aleph_0) \overset{+}{\not\to} (\aleph_0)_{144}$
- $G(\aleph_m) \overset{+}{\not\to} (\aleph_0)_{2^m \cdot 144}$ for $m < \omega$.

Monochromatic sumsets - with repetitions

$\exists\ f : \mathbb{R} \to r$ so that $f \restriction X + X$ is **not constant** for an infinite X.

Let $G(\kappa) = \bigoplus_\kappa \mathbb{Q}$ i.e. $x : \kappa \to \mathbb{Q}$ with $|supp(x)| < \omega$. E.g. $G(2^{\aleph_0}) \approx \mathbb{R}$.

[Hindman, Leader, Strauss]

- $\mathbb{Q} \overset{+}{\not\to} (\aleph_0)_{72}$.
- $G(m) \overset{+}{\not\to} (\aleph_0)_{72}$ for $m < \omega$.
- $G(\aleph_0) \overset{+}{\not\to} (\aleph_0)_{144}$
- $G(\aleph_m) \overset{+}{\not\to} (\aleph_0)_{2^m \cdot 144}$ for $m < \omega$.

Corollary

If $2^{\aleph_0} < \aleph_\omega$ then

$$\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$$

for some $r < \omega$.

Positive relations through 'position invariance'

Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.

Positive relations through 'position invariance'

Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.

Positive relations through 'position invariance'

Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.

Positive relations through 'position invariance'

Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on W** for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.

Positive relations through 'position invariance'

Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.

Positive relations through 'position invariance'

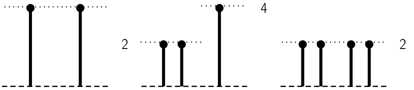
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$



$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.

Positive relations through 'position invariance'

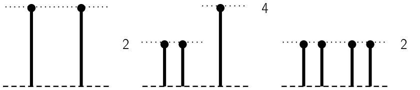
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_\kappa \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$



$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.

Positive relations through 'position invariance'

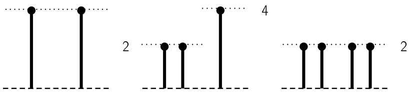
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$



$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.

Positive relations through 'position invariance'

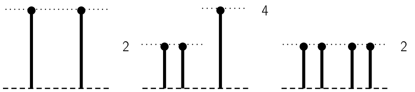
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

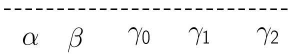


$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.



Positive relations through 'position invariance'

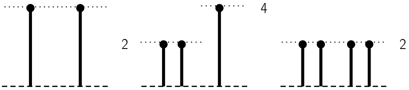
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_\kappa \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

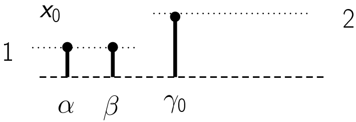


$s_0 = (4,4)$ $s_1 = (2,2,4)$ $s_2 = (2,2,2,2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

Positive relations through 'position invariance'

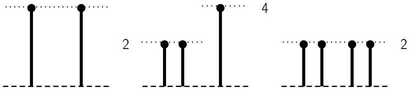
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

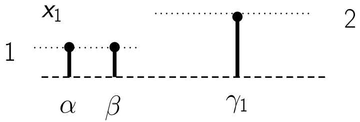


$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

Positive relations through 'position invariance'

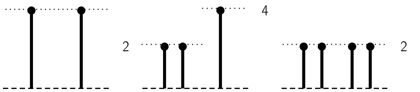
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

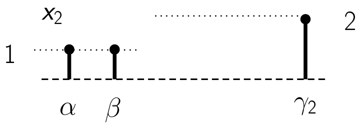


$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

Positive relations through 'position invariance'

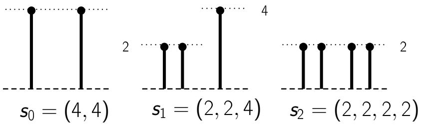
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

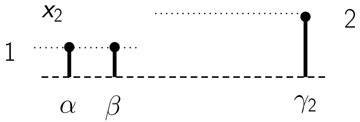
$$c_s(a) = c(s * a).$$



If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

$$c(2x_i) = c_{s_1}(a_i) = 0 = c_{s_2}(a_i \cup a_j) = c(x_i + x_j).$$

Positive relations through 'position invariance'

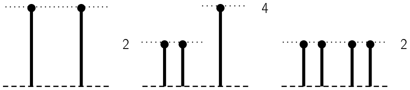
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_\kappa \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

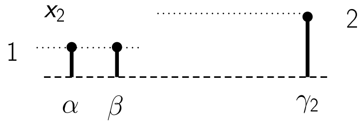


$s_0 = (4,4)$ $s_1 = (2,2,4)$ $s_2 = (2,2,2,2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

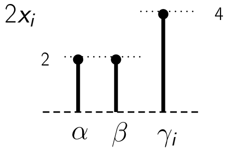
Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

$$c(2x_i) = c_{s_1}(a_i) = 0 = c_{s_2}(a_i \cup a_j) = c(x_i + x_j).$$



Positive relations through 'position invariance'

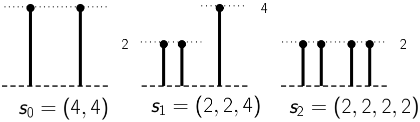
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

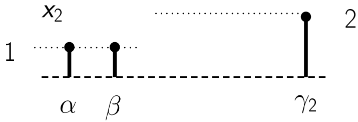
$$c_s(a) = c(s * a).$$



If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

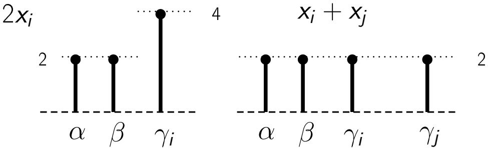
Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \cdots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

$$c(2x_i) = c_{s_1}(a_i) = 0 = c_{s_2}(a_i \cup a_j) = c(x_i + x_j).$$



Positive relations through 'position invariance'

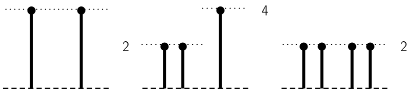
Given $s \in \mathbb{Q}^{<\omega}$ and $a \in [\kappa]^{|s|}$, let

$$x = s * a \in \bigoplus_{\kappa} \mathbb{Q}$$

by $supp(x) = a$ and $x(a(i)) = s(i)$.

Suppose that $c : \bigoplus_{\kappa} \mathbb{Q} \to 2$, and let $c_s : [\kappa]^{|s|} \to 2$ by

$$c_s(a) = c(s * a).$$

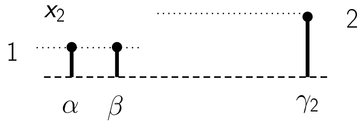


$s_0 = (4, 4)$ $s_1 = (2, 2, 4)$ $s_2 = (2, 2, 2, 2)$

If κ is large enough then there is a large $W \subseteq \kappa$ so that c_{s_i} **are constant on** W for $i = 0, 1, 2$.

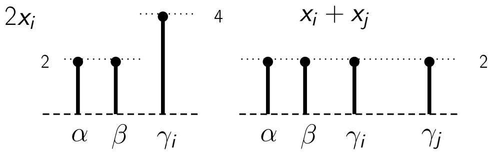
Assume that c_{s_1} and c_{s_2} are both constant 0.

Let $\alpha < \beta < \gamma_0 < \gamma_1 < \dots \in W$.



Let $a_i = \{\alpha, \beta, \gamma_i\}$ and $x_i = \frac{1}{2} s_1 * a_i$.

$$c(2x_i) = c_{s_1}(a_i) = 0 = c_{s_2}(a_i \cup a_j) = c(x_i + x_j).$$



If c_{s_0}, c_{s_2} have the same constant then we need $tp(W) = \omega + \omega$.

Corollaries

[Komjáth] and [Leader, Russell] independently

$\Rightarrow G(\kappa) \xrightarrow{+} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the Erdős-Rado theorem.

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \xrightarrow{+} (\aleph_0)_2$,

- using polarized partition relations instead.

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable, with CH and $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of [Todorcevic, di Prisco] polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and **[Leader, Russell]** independently

$\Rightarrow G(\kappa) \xrightarrow{+} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the **Erdős-Rado theorem.**

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \xrightarrow{+} (\aleph_0)_2$,

- using **polarized partition relations** instead.

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable, with **CH and** $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of **[Todorcevic, di Prisco]** polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and **[Leader, Russell]** independently

$\Rightarrow G(\kappa) \xrightarrow{+} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the **Erdős-Rado theorem.**

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \xrightarrow{+} (\aleph_0)_2$,

- using **polarized partition relations** instead.

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable, with **CH and** $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of **[Todorcevic, di Prisco]** polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and [Leader, Russell] independently

$\Rightarrow G(\kappa) \xrightarrow{+} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the Erdős-Rado theorem.

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \xrightarrow{+} (\aleph_0)_2$,

- using polarized partition relations instead.

$\Rightarrow G(\aleph_\omega) \xrightarrow{+} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable, with CH and $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of [Todorcevic, di Prisco] polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and **[Leader, Russell]** independently

$\Rightarrow G(\kappa) \stackrel{+}{\to} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \stackrel{+}{\to} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the **Erdős-Rado theorem.**

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \stackrel{+}{\to} (\aleph_0)_2$,

- using **polarized partition relations** instead.

$\Rightarrow G(\aleph_\omega) \stackrel{+}{\to} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable, with **CH and** $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of **[Todorcevic, di Prisco]** polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and **[Leader, Russell]** independently

$\Rightarrow G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \overset{+}{\not\to} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the **Erdős-Rado theorem.**

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \overset{+}{\not\to} (\aleph_0)_2$,

- using **polarized partition relations** instead.

$\Rightarrow G(\aleph_\omega) \overset{+}{\not\to} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable, with **CH and** $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of **[Todorcevic, di Prisco]** polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and **[Leader, Russell]** independently

$\Rightarrow G(\kappa) \stackrel{+}{\not\to} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \stackrel{+}{\not\to} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the **Erdős-Rado theorem.**

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \stackrel{+}{\not\to} (\aleph_0)_2$,

- using **polarized partition relations** instead.

$\Rightarrow G(\aleph_\omega) \stackrel{+}{\not\to} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable,

with **CH and** $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of **[Todorcevic, di Prisco]** polarized relation for $\aleph_\omega$ with σ-closed forcing.

Corollaries

[Komjáth] and **[Leader, Russell]** independently

$\Rightarrow G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ where $\kappa = \beth_{2r-1}(\aleph_0)$,

$\Rightarrow G(\aleph_\omega) \overset{+}{\not\to} (\aleph_0)_r$ for $r < \omega$ under GCH.

In ZFC maybe???

- using the **Erdős-Rado theorem.**

[DTS, Vidnyánszky]

$\Rightarrow G(\mathfrak{c}^+) \overset{+}{\not\to} (\aleph_0)_2$,

- using **polarized partition relations** instead.

$\Rightarrow G(\aleph_\omega) \overset{+}{\not\to} (\aleph_0)_r$ for $r < \omega$ consistently from a measurable,
with **CH and** $2^{\aleph_1} = \aleph_{\omega+1}$,

- using a version of **[Todorcevic, di Prisco]** polarized relation for $\aleph_\omega$ with σ-closed forcing.

Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_0} < \aleph_\omega$ then $\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$ for some $r < \omega$.

Consistently, modulo an ω_1-Erdős cardinal,

$$\mathbb{R} \overset{+}{\to} (\aleph_0)_r \text{ for any } r < \omega.$$

The main ingredients are

- the **position invariance** from previous proofs, but
- **polarized relations** under $MA_{\aleph_1}(Knaster)$, and
- **[S. Shelah]** Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the finite tree** $F[\bar{x}]$ for $\bar{x} \in [X]^{<\omega}$.

Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_0} < \aleph_\omega$ then $\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$ for some $r < \omega$.

Consistently, modulo an ω_1-Erdős cardinal,

$$\mathbb{R} \overset{+}{\to} (\aleph_0)_r \text{ for any } r < \omega.$$

The main ingredients are

- the **position invariance** from previous proofs, but
- **polarized relations** under $MA_{\aleph_1}(Knaster)$, and
- **[S. Shelah]** Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the finite tree** $F[\bar{x}]$ for $\bar{x} \in [X]^{<\omega}$.

Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_0} < \aleph_\omega$ then $\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$ for some $r < \omega$.

Consistently, modulo an ω_1-Erdős cardinal,

$$\mathbb{R} \overset{+}{\to} (\aleph_0)_r \textbf{ for any } r < \omega.$$

The main ingredients are

- the **position invariance** from previous proofs, but
- **polarized relations** under $MA_{\aleph_1}(Knaster)$, and
- **[S. Shelah]** Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the finite tree** $F[\bar{x}]$ for $\bar{x} \in [X]^{<\omega}$.

Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_0} < \aleph_\omega$ then $\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$ for some $r < \omega$.

Consistently, modulo an ω_1-Erdős cardinal,

$$\mathbb{R} \overset{+}{\to} (\aleph_0)_r \text{ for any } r < \omega.$$

The main ingredients are

- the **position invariance** from previous proofs, but
- **polarized relations** under $MA_{\aleph_1}(Knaster)$, and
- **[S. Shelah]** Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the finite tree** $F[\bar{x}]$ for $\bar{x} \in [X]^{<\omega}$.

Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_0} < \aleph_\omega$ then $\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$ for some $r < \omega$.

Consistently, modulo an ω_1-Erdős cardinal,

$$\mathbb{R} \overset{+}{\to} (\aleph_0)_r \text{ for any } r < \omega.$$

The main ingredients are

- the **position invariance** from previous proofs, but
- **polarized relations** under $MA_{\aleph_1}(Knaster)$, and
- [S. Shelah] Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the finite tree** $F[\bar{x}]$ for $\bar{x} \in [X]^{<\omega}$.

Positive relations on $\mathbb{R}$ - the main result

Recall: if $2^{\aleph_0} < \aleph_\omega$ then $\mathbb{R} \overset{+}{\not\to} (\aleph_0)_r$ for some $r < \omega$.

Consistently, modulo an ω_1-Erdős cardinal,

$$\mathbb{R} \overset{+}{\to} (\aleph_0)_r \text{ for any } r < \omega.$$

The main ingredients are

- the **position invariance** from previous proofs, but
- **polarized relations** under $MA_{\aleph_1}(Knaster)$, and
- **[S. Shelah]** Consistently, modulo an ω_1-Erdős cardinal, if $f : [2^{\aleph_0}]^{<\omega} \to r$ then there is an uncountable X and $F : X \hookrightarrow 2^\omega$ so that $f(\bar{x})$ **only depends on the finite tree** $F[\bar{x}]$ for $\bar{x} \in [X]^{<\omega}$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.

2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{lex} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{lex} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$
so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \upharpoonright n <_{\text{lex}} \bar{t}(l_4) \upharpoonright n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \upharpoonright m <_{\text{lex}} \bar{s}(l_4) \upharpoonright m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$
so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.

2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$
  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

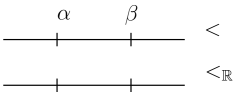
  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.



2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$ iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

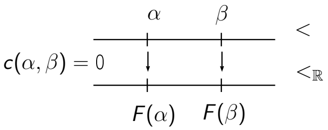
  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$
so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.



2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$
  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

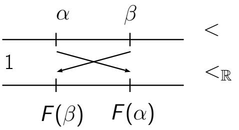
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$
so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \upharpoonright n <_{\text{lex}} \bar{t}(l_4) \upharpoonright n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \upharpoonright m <_{\text{lex}} \bar{s}(l_4) \upharpoonright m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

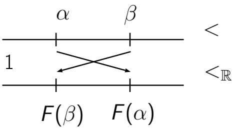
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

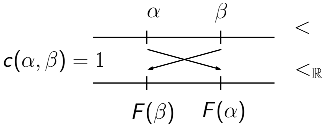
  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$
so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.



2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,

- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$
  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,

- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

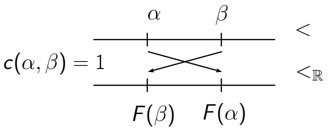
  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$
so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.



2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$
  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

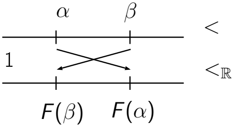
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



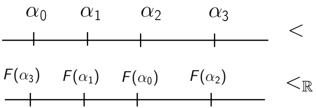
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{lex} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{lex} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

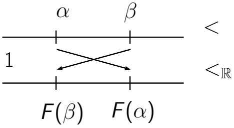
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

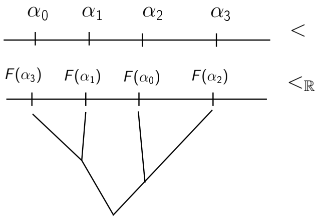
Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.

$c(\alpha, \beta) = 1$





2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

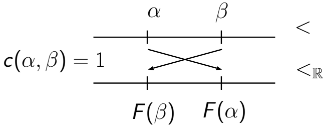
  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.



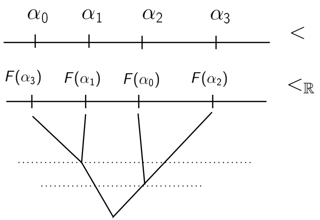
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{lex} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{lex} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

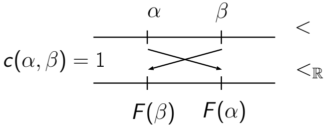
  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.



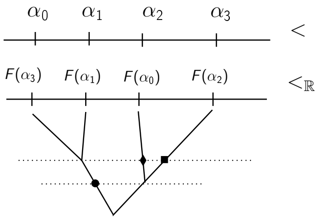
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,

- $\bar{t}(l_3) \restriction n <_{lex} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{lex} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,

- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

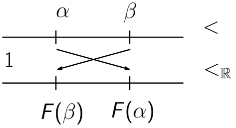
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



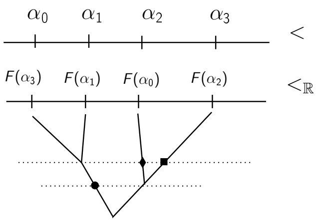
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: $\Delta''[X]^2$ **is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{lex} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{lex} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

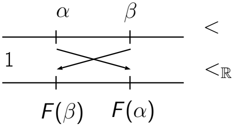
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



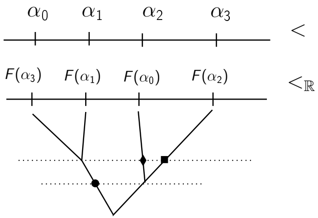
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$ iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$ iff $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$ iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

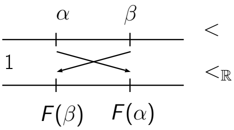
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



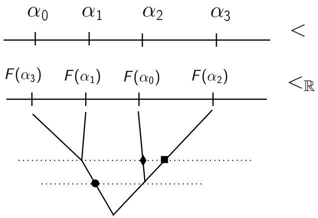
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

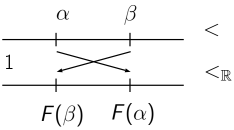
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$.

$c(\alpha, \beta) = 1$



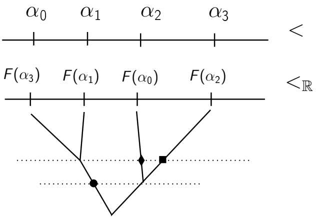
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^\omega$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \restriction n <_{\text{lex}} \bar{t}(l_4) \restriction n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \restriction m <_{\text{lex}} \bar{s}(l_4) \restriction m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

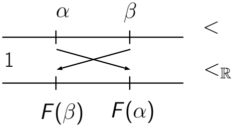
Positive partition relations on $\kappa = 2^{\aleph_0}$? No way...

Sierpinski colouring: $c : [2^{\aleph_0}]^2 \to 2$ so that $c(\alpha, \beta) = 0$ iff

$$\alpha < \beta \leftrightarrow F(\alpha) <_{\mathbb{R}} F(\beta)$$

for some fixed $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$.

$c(\alpha, \beta) = 1$



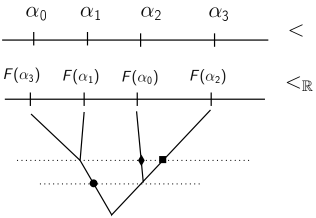
2 colours on any uncountable set!

You can define more complicated

$$c : [2^{\aleph_0}]^k \to r$$

using an $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^{\omega}$ and the values of

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$



Also: **$\Delta''[X]^2$ is infinite** for any infinite $X \subseteq 2^{\omega}$.

Say $\bar{t} \sim \bar{s}$ for $\bar{s}, \bar{t} \in \mathbb{R}^i$ iff for all $l_1, l_2, l_3, l_4 < i$:

- $\Delta(\bar{t}(l_1), \bar{t}(l_2)) < \Delta(\bar{t}(l_3), \bar{t}(l_4))$

  iff $\Delta(\bar{s}(l_1), \bar{s}(l_2)) < \Delta(\bar{s}(l_3), \bar{s}(l_4))$,
- $\bar{t}(l_3) \upharpoonright n <_{\text{lex}} \bar{t}(l_4) \upharpoonright n$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff

  $\bar{s}(l_3) \upharpoonright m <_{\text{lex}} \bar{s}(l_4) \upharpoonright m$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$,
- $\bar{t}(l_3)(n) = 0$ for $n = \Delta(\bar{t}(l_1), \bar{t}(l_2))$

  iff $\bar{s}(l_3)(m) = 0$ for $m = \Delta(\bar{s}(l_1), \bar{s}(l_2))$.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then c **is F-canonical on** $W \subseteq 2^{\aleph_0}$ iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i **is F-canonical on** W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]
Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

⋆

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]
Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is **$W \in [\lambda]^{\aleph_1}$ and $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$** so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]
Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]
Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is **$W \in [\lambda]^{\aleph_1}$ and $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$** so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]
Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]
Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is **$W \in [\lambda]^{\aleph_1}$ and $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$** so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then c **is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i **is F-canonical on W.**

Polarized partition relation in this model: $MA_{\aleph_1}(\text{Knaster}) \Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]
Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]
Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]
Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

⋆

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]
Suppose that λ is an ω_1-Erdős cardinal in V.
Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]
Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]
Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}(\text{Knaster}) \Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is **$W \in [\lambda]^{\aleph_1}$ and $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$** so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar{\alpha}) = c(\bar{\beta})$ whenever $F(\bar{\alpha}) \sim F(\bar{\beta})$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$, then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then c **is** F**-canonical on** $W \subseteq 2^{\aleph_0}$ iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is $W \in [\lambda]^{\aleph_1}$ **and** $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$ so that

c_i **is** F**-canonical on** W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Shelah's 'Was Sierpinski right?' papers

We cannot realize more colours:

[Shelah, WSR I]

Consistently, modulo an ω_1-Erdős cardinal, **for any $c : [2^{\aleph_0}]^2 \to r$ there is an uncountable $W \subseteq 2^{\aleph_0}$ with at most 2 colours.**

$\star$

Larger tuples can define more colours... What was so specific about the colourings before?

if $c : [2^{\aleph_0}]^k \to r$ and $F : 2^{\aleph_0} \hookrightarrow \mathbb{R} \simeq 2^\omega$

then **c is F-canonical on $W \subseteq 2^{\aleph_0}$** iff

$c(\bar\alpha) = c(\bar\beta)$ whenever $F(\bar\alpha) \sim F(\bar\beta)$.

[Shelah, WSR II]

Suppose that λ is an ω_1-Erdős cardinal in V.

Then there is a forcing notion $\mathbb{P}$ so that $V^{\mathbb{P}}$ satisfies the following:

- $2^{\aleph_0} = \lambda$,
- $MA_{\aleph_1}$(Knaster), and

if $c_i : [\lambda]^i \to r$ for $i < k < \omega, r < \omega$,
then there is **$W \in [\lambda]^{\aleph_1}$ and $F : W \hookrightarrow \mathbb{R} \simeq 2^\omega$** so that

c_i is F-canonical on W.

Polarized partition relation in this model: $MA_{\aleph_1}$(Knaster) $\Rightarrow$ if $g : [\omega]^2 \times \omega_1 \to 2$ then there is $A \in [\omega]^\omega, B \in [\omega_1]^{\omega_1}$ so that $g \restriction [A]^2 \times B$ is constant.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.

$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \overset{+}{\to} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.

$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that **c_{s_i} is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.

$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.

$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

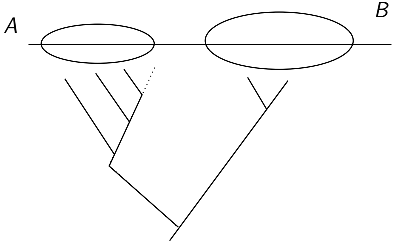
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \overset{+}{\to} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that **c_{s_i} is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so **c_{s_0} is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!
$\Rightarrow$ **c_{s_1} is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.
$\Rightarrow$ **c_{s_2} is constant** on these 4-tuples.

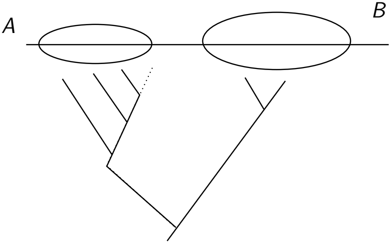
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that **c_{s_i} is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so **c_{s_0} is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ **c_{s_1} is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ **c_{s_2} is constant** on these 4-tuples.

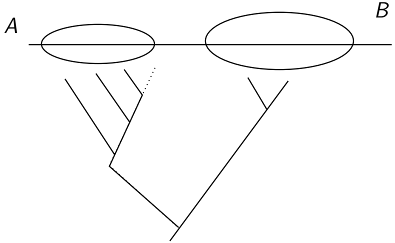
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

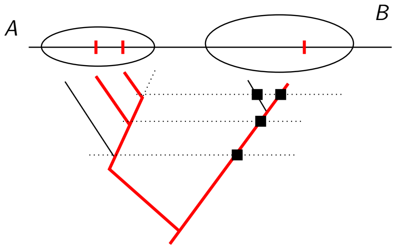
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \stackrel{+}{\to} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

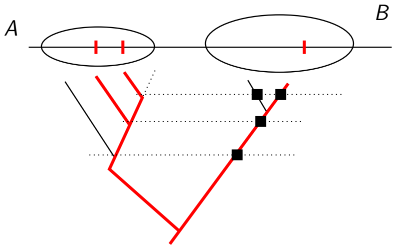
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

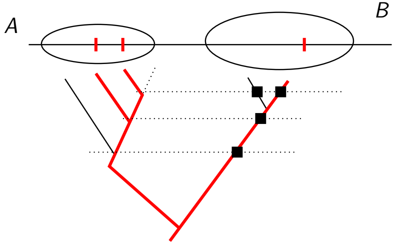
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

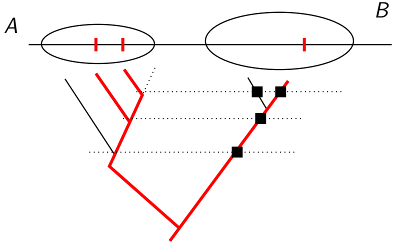
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \overset{+}{\to} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

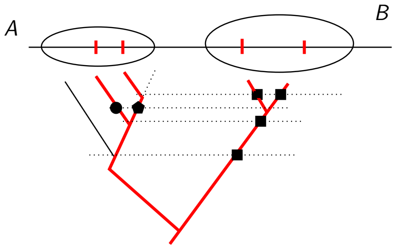
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values.

This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

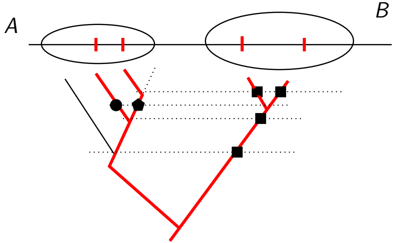
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

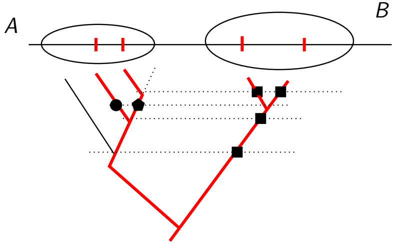
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

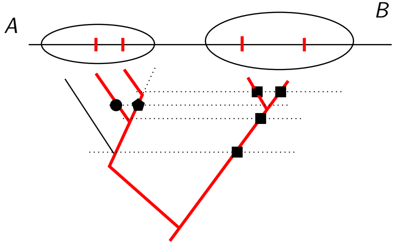
Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Proving $G(2^{\aleph_0}) \xrightarrow{+} (\aleph_0)_2$ in the 'WSR II' model

Take $c : G(2^{\aleph_0}) \to 2$ and consider $c_{s_0}, c_{s_1}, c_{s_2}$ with $s_0 = (4,4), s_1 = (2,2,4), s_2 = (2,2,2,2)$.

Apply WSR II: there is $|W| = \aleph_1$ and $F : W \hookrightarrow \mathbb{R}$ so that c_{s_i} **is F-canonical on W**.

Select $|A| = \aleph_0, |B| = \aleph_1$ from W so that $A < B$ and $F''A <_{\mathbb{R}} F''B$.



$\Rightarrow$ all pairs $(\alpha, \beta) \in A \times B$ have the same $\sim$-type, so c_{s_0} **is constant**.

How can we fix the type of triples

$$(\alpha, \alpha', \beta) \in A^2 \times B?$$

Let $g : A^2 \times B \to 2$ by

$$g(\alpha, \alpha', \beta) = F(\beta)(m)$$

with $m = \Delta(F(\alpha), F(\alpha'))$. Shrink using the polarized relation to fix the type!

$\Rightarrow$ c_{s_1} **is constant** too on these triples.

Finally, look at 4-tuples

$$(\alpha, \alpha', \beta, \beta') \in A^2 \times B^2.$$

Look at splitting levels from B, read values on branches from A, thin both to fix the values. This fixes the type of these 4-tuples too on some countable A, B.

$\Rightarrow$ c_{s_2} **is constant** on these 4-tuples.

Now, **two constant values must agree** of the three; repeat the first trick to construct infinite X so that $X + X$ is monochromatic.

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \dots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- **$G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ for $r < \omega$ in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- **$G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ for $r < \omega$ in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- **$G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ for $r < \omega$ in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- **$G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ for $r < \omega$ in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- **$G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ for $r < \omega$ in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???

Various open problems – Thank you for your attention!

[Owings, 1974]

- $\mathbb{N} \overset{+}{\not\to} (\aleph_0)_2$???

Connected to our results:

- $\mathbb{R} \overset{+}{\to} (\aleph_0)_2$ in ZFC??
- $G(\aleph_\omega) \overset{+}{\to} (\aleph_0)_r$ **for** $r < \omega$ **in ZFC???**
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ if $2^{\aleph_0}$ is real-valued measurable?
- $\mathbb{R} \overset{+}{\to} (\aleph_0)_r$ without large cardinals?
- unbalanced sumsets $X + Y$ (or $X \oplus Y$)?

What is the smallest r so that $G(\kappa) \overset{+}{\not\to} (\aleph_0)_r$ for a particular κ (finite, or $\aleph_m$)??

Monochromatic k-sumsets: $X + X + \cdots + X$?

[HLS] There is a finite colouring of $G(\aleph_n)$ with no infinite monochromatic k-sumsets ($n < \omega$),

[DS, Vidnyánszky] There is a finite coloring of $\mathbb{R}$ with no infinite monochromatic k-sumsets for $k \geq 3$.

We are far from a complete picture.

[Shelah, 1988]

- Is $2^{\aleph_0} = \aleph_m \to [\aleph_1]^2_3$ consistent for some $m < \omega$???
- Is $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$ consistent???