# Monochromatic sumsets for colourings of $\mathbb{R}$ 

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## Introduction

Consistently, modulo some large cardinal,
if $f: \mathbb{R} \rightarrow r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{R}$ so that $f \upharpoonright X+X$ is constant.
$X+X=\{x+y: x, y \in X\}$ i.e. repetitions are allowed.

- How does this fit into the theory of partition relations?
- What goes into the proof of this result?
- Joint result with P. Komjáth, I. Leader, P. Russell, S. Shelah, D. T Soukup, and Z. Vidnyánszky.


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## Evolving partition relations

... an incomplete overview ...

## Evolving partition relations

If $f: \omega \rightarrow r$ then there is an infinite $X \subset \omega$ with $f \upharpoonright X$ constant.
P. H. Principle

$$
\omega \rightarrow(\omega)_{r}^{1}
$$

## Evolving partition relations

If $f:[\omega]^{k} \rightarrow r$ then there is an infinite $X \subset \omega$ with $f \upharpoonright[X]^{k}$ constant.
P. H. Principle

$$
\omega \rightarrow(\omega)_{r}^{1}
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## Evolving partition relations

There is $f:\left[2^{\aleph_{0}}\right]^{2} \rightarrow 2$ so that $f^{\prime \prime}[X]^{2}=2$ for any uncountable $X \subset 2^{\aleph_{0}}$.
P. H. Principle

$$
\omega \rightarrow(\omega)_{r}^{1}
$$



## Evolving partition relations

$$
\text { If } f:\left[\beth_{k-1}^{+}\right]^{k} \rightarrow r \text { then } f \upharpoonright[W]^{k} \text { is constant for some uncountable } W \subseteq \beth_{k-1}^{+} \text {. }
$$

P. H. Principle

$$
\omega \rightarrow(\omega)_{r}^{1}
$$

W. Sierpinski, 1933

$$
2^{\aleph_{0}} \nrightarrow\left(\aleph_{1}\right)_{2}^{2}
$$

Erdős, Rado 1956

$$
\beth_{k-1}^{+} \rightarrow\left(\omega_{1}\right)_{r}^{k} \text { for all } r<\omega
$$

## Evolving partition relations

FinSum $(X)=\left\{x_{0}+x_{1}+\cdots+x_{\ell}: x_{0}<\cdots<x_{\ell} \in X\right\}$ i.e. no repetitions.
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$\beth_{k-1}^{+} \rightarrow\left(\omega_{1}\right)_{r}^{k}$ for all $r<\omega$.
N. Hindman, 1974
if $f: \mathbb{N} \rightarrow r$ then there is some infinite $X \subseteq \mathbb{N}$ so that $f \upharpoonright \operatorname{FinSum}(X)$ is constant.

## Evolving partition relations

There is $f:\left[\aleph_{1}\right]^{2} \rightarrow \aleph_{1}$ so that $f^{\prime \prime}[X]^{2}=\aleph_{1}$ for any uncountable $X \subset \aleph_{1}$.
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F. P. Ramsey, 1930

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S. Todorcevic, 1987

$$
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## Evolving partition relations

## If $f:\left[2^{\aleph_{0}}\right]^{2} \rightarrow 3$ then there is an uncountable $X \subset 2^{\aleph_{0}}$ with $\left|f^{\prime \prime}[X]^{2}\right| \leq 2$.



## Monochromatic sumsets in $\mathbb{N}$

## Easy Ramsey consequence: if $f: \mathbb{N} \rightarrow r$ with $r \in \omega$ then there is an infinite $X \subseteq \mathbb{N}$ so that

$$
f \upharpoonright X \oplus X \text { is constant }
$$

## Here $X \oplus X=\{x+y: x \neq y \in X\}$ i.e. repetitions are not allowed.

 Proof:[Owings, Hindman 1970s] What happens if we allow repetition?

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- if $f: \mathbb{N} \rightarrow r$ then let $g:[\mathbb{N}]^{2} \rightarrow r$ defined by $g(x, y)=f(x+y)$,
- if $X \subset \mathbb{N}$ and $g \upharpoonright[X]^{2}$ is constant then $f \upharpoonright X \oplus X$ is constant too.
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## Monochromatic sumsets in $\mathbb{N}$ - with repetitions?

$X+X=X \oplus X \cup\{2 x: x \in X\}$.

## There is $f: \mathbb{N} \rightarrow 4$ without infinite monochromatic sumsets:



- Suppose that $X \subseteq \mathbb{N}$ is infinite and take $y \ll x \in X$.
- $\left|\log _{\sqrt{2}}(x)-\log _{\sqrt{2}}(x+y)\right|<1$.
- $|f(x)-f(x+y)| \leq 1 \bmod 4$.
- $f(2 x)=\left\lfloor\log _{\sqrt{2}}(x)+2\right\rfloor=f(x)+2 \bmod 4$ so $f(2 x) \neq f(x+y)$.
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Can we do this with 2 colours???


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## Monochromatic sumsets in $\mathbb{R}$

## Started in [Hindman, Leader, Strauss 2015] <br> If $f: \mathbb{R} \rightarrow r$ is Baire/Lebesgue measurable then there is a perfect $\emptyset \neq X \subseteq \mathbb{R}$ so that

## $f \upharpoonright X+X$ is constant

## Without definability?

There is an $f: \mathbb{R} \rightarrow 2$ so that

$$
f^{\prime \prime} X \oplus X=2 \text { for every uncountable } X \subset \mathbb{R} .
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- [HLS] using CH, [Komjáth, DTS, Weiss] in ZFC, and consistently the number of colours is best possible.


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## Monochromatic sumsets in $\mathbb{R}$

## Continued by [Fernandez-Breton, Rinot 2016]: <br> - how to realize more colours on sets of the form FinSum $(X)$ (no repetitions), <br> - general theorems on uncountable, commutative, cancellative semigroups $G$.

## Bottom line: without definabilty,

Infinite sumsets are best possible on $\mathbb{R}$ with repetition allowed.

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## Monochromatic sumsets - with repetitions

Recall: $\exists f: \mathbb{N} \rightarrow 4$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset \mathbb{N}$.

## Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.

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## Monochromatic sumsets - with repetitions

If $f: G(\kappa) \rightarrow r$ then $f \upharpoonright X+X$ is constant for some infinite $X \subset G(\kappa)$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$. Notation:

$$
G(\kappa) \xrightarrow{+}\left(\aleph_{0}\right)_{r}
$$

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$$
G(\kappa) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{r} \text { e.g. } \stackrel{\mathbb{N}}{\stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{4}}
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## Monochromatic sumsets - with repetitions

$\exists f: \mathbb{Q} \rightarrow 72$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset \mathbb{Q}$.

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- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.


## Monochromatic sumsets - with repetitions

$\exists f: G(m) \rightarrow 72$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset G(m)$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
[Hindman, Leader, Strauss]

- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.
- $G(m) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$ for $m<\omega$.


## Monochromatic sumsets - with repetitions

$\exists f: G\left(\aleph_{0}\right) \rightarrow 144$ so that $f \upharpoonright X+X$ is not constant for an infinite $X \subset G\left(\aleph_{0}\right)$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
[Hindman, Leader, Strauss]

- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.
- $G(m) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$ for $m<\omega$.
- $G\left(\aleph_{0}\right) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{144}$


## Monochromatic sumsets - with repetitions

$\exists f: G\left(\aleph_{m}\right) \rightarrow 2^{m} \cdot 144$ so that $f \upharpoonright X+X$ is not constant for an infinite $X$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
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- $G\left(\aleph_{m}\right) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{2^{m} \cdot 144}$ for $m<\omega$.


## Monochromatic sumsets - with repetitions

$\exists f: \mathbb{R} \rightarrow r$ so that $f \upharpoonright X+X$ is not constant for an infinite $X$.

Let $G(\kappa)=\bigoplus_{\kappa} \mathbb{Q}$ i.e. $x: \kappa \rightarrow \mathbb{Q}$ with $|\operatorname{supp}(x)|<\omega$. E.g. $G\left(2^{\aleph_{0}}\right) \approx \mathbb{R}$.
[Hindman, Leader, Strauss]

- $\mathbb{Q} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$.
- $G(m) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{72}$ for $m<\omega$.
- $G\left(\aleph_{0}\right) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{144}$
- $G\left(\aleph_{m}\right) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{2^{m} \cdot 144}$ for $m<\omega$.


## Corollary

If $2^{\aleph_{0}}<\aleph_{\omega}$ then
$\mathbb{R} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{r}$
for some $r<\omega$.

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Given }s\in\mp@subsup{\mathbb{Q}}{}{<\omega}\mathrm{ and }a\in[\kappa\mp@subsup{]}{}{|s|}\mathrm{ , let
    by }\operatorname{supp}(x)=a\mathrm{ and }x(a(i))=s(i)
Suppose that c:(D) (O) }->2\mathrm{ , and let
cs:[k] }\mp@subsup{}{}{s}->2\mathrm{ by
cs(a) =c(s*a).
by \(\operatorname{supp}(x)=a\) and \(x(a(i))=s(i)\).
Suppose that \(c:(0) \rightarrow 2\) and let \(c_{s}:[\kappa]^{s} \rightarrow 2\) by \(c_{s}(a)=c(s * a)\).
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Assume that $c_{s_{1}}$ and $c_{s_{2}}$ are both constant 0 .

## Positive relations through 'position invariance'

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If \(c_{s_{0}}, c_{s_{2}}\) have the same constant then we need \(\operatorname{tp}(W)=\omega+\omega\).

\section*{Corollaries}

\section*{[Komjáth] and [Leader, Russell] independently}
\(\Rightarrow G(\kappa) \xrightarrow{+}\left(\aleph_{0}\right)_{r}\) where \(\kappa=\beth_{2 r-1}\left(\aleph_{0}\right)\),

- using the Erdős-Rado theorem.

\section*{[DTS, Vidnyánszky]}

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\section*{Positive relations on \(\mathbb{R}\) - the main result}

Recall: if \(2^{\aleph_{0}}<\aleph_{\omega}\) then \(\mathbb{R} \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{r}\) for some \(r<\omega\).

\section*{Consistently, modulo an \(\omega_{1}\)-Erdős cardinal,}
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\section*{The main ingredients are}
- the position imvariance from previous proofs, but
- polarized relations under \(M A_{\aleph_{1}}\) (Knaster), and
- [S. Shelah] Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, if \(f:\left[2^{\aleph_{0}}\right]^{<\omega} \rightarrow r\) then there is an uncountable \(X\) and \(F: X \hookrightarrow 2^{\omega}\) so that \(f(\bar{x})\) only depends on the finite tree \(F[\bar{x}]\) for \(\bar{x} \in[X]^{<\omega}\).

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- [S. Shelah] Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, if \(f:\left[2^{\aleph_{0}}\right]^{<\omega} \rightarrow r\) then there is an uncountable \(X\) and \(F: X \hookrightarrow 2^{\omega}\) so
that \(f(\bar{x})\) only depends on the finite tree \(F[\bar{x}]\) for \(\bar{x} \in[X]^{<\omega}\).

\section*{Positive relations on \(\mathbb{R}\) - the main result}

Recall: if \(2^{\aleph_{0}}<\aleph_{\omega}\) then \(\mathbb{R} \stackrel{+}{\rightarrow}\left(\aleph_{0}\right)_{r}\) for some \(r<\omega\).

Consistently, modulo an \(\omega_{1}\)-Erdős cardinal,
\[
\mathbb{R} \xrightarrow{+}\left(\aleph_{0}\right)_{r} \text { for any } r<\omega
\]

The main ingredients are
- the position invariance from previous proofs, but
- polarized relations under \(M A_{\aleph_{1}}\) (Knaster), and
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Positive partition relations on \(\kappa=2^{\aleph_{0}}\) ? No way...
```

Sierpinski colouring: c: [2 \
so that c(\alpha,\beta)=0 iff
\alpha<\beta\leftrightarrowF(\alpha)<\mp@subsup{\mathbb{R}}{}{\prime}F(\beta)
for some fixed F: 2 N

```
    Also: \(\Delta^{\prime \prime}[X]^{2}\) is infinite for any infinite \(X \subseteq 2^{\omega}\).
    Sav \(\bar{t} \sim \bar{s}\) for \(\bar{s}, \bar{t} \in \mathbb{R}^{i}\) iff for all \(I_{1}, l_{2}, l_{3}, I_{A}<i\) :
2 colours on any uncountable set!
You can define more complicated
```

        c:[2\mp@subsup{N}{0}{}}\mp@subsup{]}{}{k}->
    using an F: 2N0}\hookrightarrow\mathbb{R}\simeq\mp@subsup{2}{}{\omega}\mathrm{ and the
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\Delta(x,y)=\operatorname{min}{n:x(n)\not=y(n)}.

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\section*{Shelah's 'Was Sierpinski right?' papers}
```

We cannot realize more colours:
[Shelah, MISR 1]
Consistently, modulo an }\mp@subsup{\omega}{1}{}\mathrm{ -Erdős cardinal, for
any c:[2\mp@subsup{N}{0}{}}\mp@subsup{]}{}{2}->r\mathrm{ there is an uncountable
W \subset 2 ^ { \aleph _ { 0 } } with at most 2 colours.
Larger tuples can define more colours... What
was so specific about the colourings before?
if c:[2N0}\mp@subsup{]}{}{k}->r\mathrm{ and }F:\mp@subsup{2}{}{\mp@subsup{N}{0}{}}\hookrightarrow\mathbb{R}\simeq\mp@subsup{2}{}{\omega
c(\overline{\alpha})=c(\overline{\beta})\mathrm{ whenever }F(\overline{\alpha})~F(\overline{\beta}).

```
then \(c\) is \(F\)-canonical on \(v v \subset 20\) iff \(c_{i}\) is \(F\)-canonical on \(W\).

\section*{[Shelah, WSR II]}

Suppose that \(\lambda\) is an \(\omega_{1}\)-Erdös cardinal in \(V\)
Then there is a forcing notion \(\mathbb{P}\) so that \(V^{\mathbb{P}}\) satisfies the following:
- \(2^{\aleph_{0}}=\lambda\),
- \(\mathrm{MA}_{\mathrm{N}_{1}}\) (Knaster), and
if \(c_{i}:[\lambda]^{i} \rightarrow r\) for \(i<k<\omega, r<\omega\), then there is \(W \in[\lambda]^{\aleph_{1}}\) and \(F: W \hookrightarrow \mathbb{R} \simeq 2^{\omega}\) so that

Polarized partition relation in this model: \(\mathrm{MA}_{\aleph_{1}}\) (Knaster) \(\Rightarrow\) if \(g:[\omega]^{2} \times \omega_{1} \rightarrow 2\) then there is \(A \in[\omega]^{\omega}, B \in\left[\omega_{1}\right]^{\omega_{1}}\) so that \(g \upharpoonright[A]^{2} \times B\) is constant.

\section*{Shelah's 'Was Sierpinski right?' papers}

We cannot realize more colours:
[Shelah, WSR I]
Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, for any \(c:\left[2^{\aleph_{0}}\right]^{2} \rightarrow r\) there is an uncountable \(W \subset 2^{\kappa_{0}}\) with at most 2 colours.

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if \(c:\left[2^{N_{0}}\right]^{k} \rightarrow r\) and \(F: 2^{N_{0}} \hookrightarrow \mathbb{R} \simeq 2^{\omega}\)
\(c_{i}\) is \(F\)-canonical on \(W\).
\(c(\bar{\alpha})=c(\bar{\beta})\) whenever \(F(\bar{\alpha}) \sim F(\bar{\beta})\).

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then \(c\) is \(F\)-canonical on \(W \subset 2^{\aleph_{0}}\) iff
\(c^{\prime}(\bar{\alpha})=c(\bar{\beta})\) whenever \(F(\bar{\alpha}) \sim F(\bar{\beta})\)

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\section*{Shelah's 'Was Sierpinski right?' papers}

We cannot realize more colours:
[Shelah, WSR I]
Consistently, modulo an \(\omega_{1}\)-Erdős cardinal, for any \(c:\left[2^{\aleph_{0}}\right]^{2} \rightarrow r\) there is an uncountable \(W \subseteq 2^{\aleph_{0}}\) with at most 2 colours.
\(\star\)

Larger tuples can define more colours... What was so specific about the colourings before?
if \(c:\left[2^{\aleph_{0}}\right]^{k} \rightarrow r\) and \(F: 2^{N_{0}} \hookrightarrow \mathbb{R} \simeq 2^{\omega}\)
then \(c\) is \(F\)-canonical on \(W \subseteq 2^{\aleph_{0}}\) iff
\(c(\bar{\alpha})=c(\bar{\beta})\) whenever \(F(\bar{\alpha}) \sim F(\bar{\beta})\).

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Suppose that \(\lambda\) is an \(\omega_{1}\)-Erdős cardinal in \(V\)
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\section*{Proving \(G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}\) in the 'WSR II' model}
```

Take c:G(2 (
with sol}=(4,4),\mp@subsup{s}{1}{}=(2,2,4),\mp@subsup{s}{2}{}=(2,2,2,2)
Apply WSR II: there is |W|=\mp@subsup{\aleph}{1}{}\mathrm{ and}
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```
Select \(|A|=\aleph_{0},|B|=\aleph_{1}\) from \(W\) so that
\(A<B\) and \(F^{\prime \prime} A<\mathbb{R} F^{\prime \prime} B\).
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How can we fix the type of triples
(\alpha,\mp@subsup{\alpha}{}{\prime},\beta)\in\mp@subsup{A}{}{2}\timesB?
Let g: A }\mp@subsup{A}{}{2}\timesB->2\mathrm{ by
g(\alpha,\mp@subsup{\alpha}{}{\prime},\beta)=F(\beta)(m)
with m=\Delta(F(\alpha),F(\mp@subsup{\alpha}{}{\prime})). Shrink using the
polarized relation to fix the type!
C}\mp@subsup{c}{\mp@subsup{S}{1}{}}{}\mathrm{ is constant too on these triples.
Finally, look at 4-tuples
(\alpha,\mp@subsup{\alpha}{}{\prime},\beta,\mp@subsup{\beta}{}{\prime})\in\mp@subsup{A}{}{2}\times\mp@subsup{B}{}{2}.
Look at splitting levels from B, read values on
branches from A, thin both to fix the values.
This fixes the type of these 4-tuples too on
some countable A, B.
C}\mp@subsup{C}{\mp@subsup{S}{2}{}}{}\mathrm{ is constant on these 4-tuples.
Now, two constant values must agree of the
three; repeat the first trick to construct infinite
X so that }X+X\mathrm{ is monochromatic.

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\section*{Proving \(G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}\) in the 'WSR II' model}

Take \(c: G\left(2^{\aleph_{0}}\right) \rightarrow 2\) and consider \(c_{s_{0}}, c_{s_{1}}, c_{s_{2}}\) with \(s_{0}=(4,4), s_{1}=(2,2,4), s_{2}=(2,2,2,2)\). Apply WSR II: there is \(|W|=\aleph_{1}\) and \(F: W \hookrightarrow \mathbb{R}\) so that \(c_{s_{i}}\) is \(F\)-canonical on \(W\) Select \(|A|=\aleph_{0},|B|=\aleph_{1}\) from \(W\) so that \(A<B\) and \(F^{\prime \prime} A<_{\mathbb{R}} F^{\prime \prime} B\).

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Look at splitting levels from \(B\), read values on
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This fixes the type of these 4-tuples too on
some countable \(A, B\).
\(\Rightarrow C_{s_{2}}\) is constant on these 4-tuples.

Now, two constant values must agree of the three; repeat the first trick to construct infinite \(X\) so that \(X+X\) is monochromatic.

\section*{Proving \(G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}\) in the 'WSR II' model}

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g\left(\alpha, \alpha^{\prime}, \beta\right)=F(\beta)(m)
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with \(m=\Delta\left(F(\alpha), F\left(\alpha^{\prime}\right)\right)\). Shrink using the polarized relation to fix the type! \(\Rightarrow c_{s_{1}}\) is constant too on these triples.

Finally, look at 4-tuples

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\section*{Proving \(G\left(2^{\aleph_{0}}\right) \xrightarrow{+}\left(\aleph_{0}\right)_{2}\) in the 'WSR II' model}

Take \(c: G\left(2^{\aleph_{0}}\right) \rightarrow 2\) and consider \(c_{s_{0}}, c_{s_{1}}, c_{s_{2}}\) with \(s_{0}=(4,4), s_{1}=(2,2,4), s_{2}=(2,2,2,2)\).
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Select \(|A|=\aleph_{0},|B|=\aleph_{1}\) from \(W\) so that \(A<B\) and \(F^{\prime \prime} A<\mathbb{R} F^{\prime \prime} B\).

\(\Rightarrow\) all pairs \((\alpha, \beta) \in A \times B\) have the same \(\sim\)-type, so \(c_{s_{0}}\) is constant.

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\section*{Various open problems}

\section*{[Owings, 1974]}
- \(\mathbb{N} / 4\left(\mathrm{~N}_{0}\right)_{2}\) ???

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\section*{Monochromatic \(k\)-sumsets: \(X+X+\cdots+X\) ?}
[HLS] There is a finite colouring of \(G\left(\aleph_{n}\right)\) with no infinite monochromatic \(k\)-sumsets ( \(n<\omega\) ),
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We are far from a complete picture.
[Shelah, 1988]
- Is \(2 \mathrm{~N}_{\mathrm{n}}-\mathrm{N} \rightarrow\left[\mathrm{N}_{1}\right]_{3}^{2}\) consistent
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\text { for some } m<\omega ? ? ?
\]
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Monochromatic \(k\)-sumsets: \(X+X+\cdots+X\) ?
[HLS] There is a finite colouring of \(G\left(\aleph_{n}\right)\) with no infinite monochromatic \(k\)-sumsets \((n<\omega)\),
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We are far from a complete picture.
[Shelah, 1988]
- Is \(2 N_{0}-N \rightarrow\left[N_{1}\right]_{3}^{2}\) consistent
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What is the smallest \(r\) so that \(G(\kappa) \stackrel{+}{\nrightarrow}\left(\aleph_{0}\right)_{r}\) for a particular \(\kappa\) (finite, or \(\aleph_{m}\) )??

\section*{Various open problems}
[Owings, 1974]
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