Orientations of graphs with uncountable chromatic number

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Introduction

Goal: present results on **chromatic number of directed uncountable graphs**.

- first organized effort (undirected case): P. Erdős and A. Hajnal in the 1960s;
- significant contributions: P. Komjáth, S. Todorcevic, S. Shelah, C. Thomassen...

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*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*

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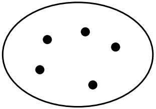

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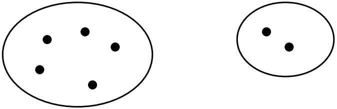

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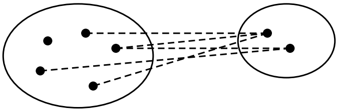

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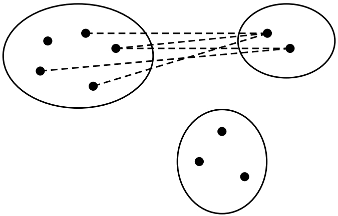

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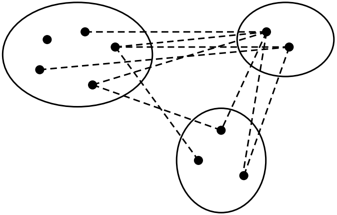

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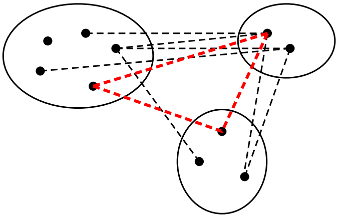

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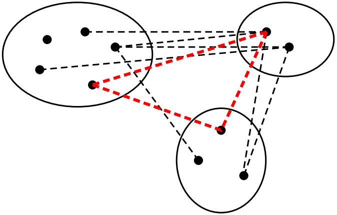

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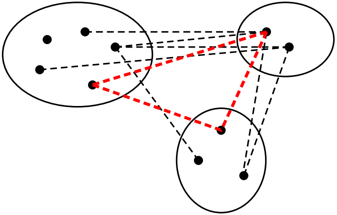

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Photograph by J. A. Bondy.

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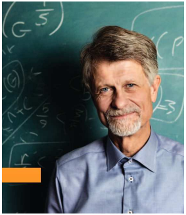

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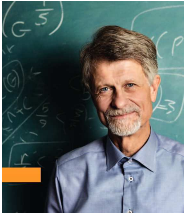

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Shift graphs

Define the **shift graph** $\mathrm{Sh}_n(\kappa)$ (for $2 \leq n < \omega$) on $[\kappa]^n$ by connecting

$$u = \{\xi_0 < \cdots < \xi_{n-1}\} \text{ with } v = \{\xi_1 < \cdots < \xi_n\}.$$

If κ is large enough then **the chromatic number of $\mathrm{Sh}_n(\kappa)$ is large**.

No odd cycles of length $\leq 2n - 1$.

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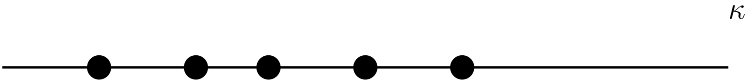

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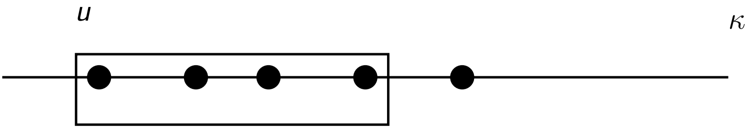

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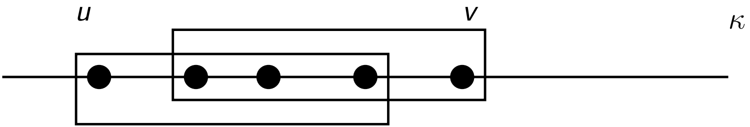

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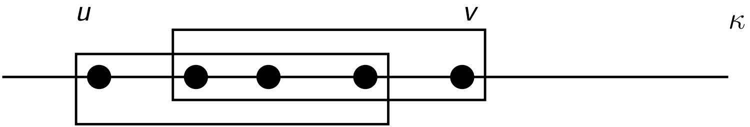

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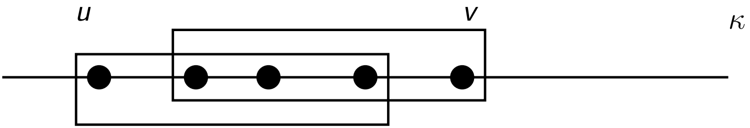

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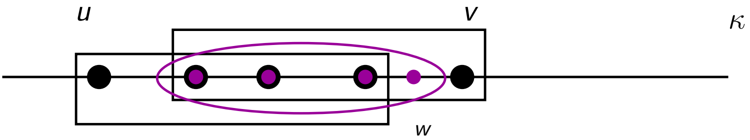

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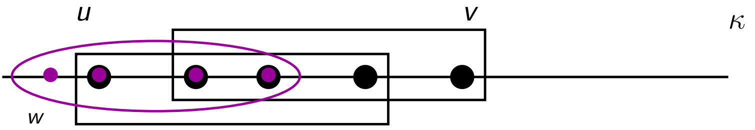

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The dichromatic number of digraphs

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The dichromatic number of a digraph D, denoted by $\overrightarrow{\chi}(D)$, is the least cardinal κ such that the vertices of D can be covered by κ many acyclic sets.

What are the implications of large dichromatic number? How is $\overrightarrow{\chi}(D)$ related to the chromatic number of the underlying graph?

If $\overrightarrow{\chi}(D) \geq \kappa$ then the underlying directed graph must have chromatic number $\geq \kappa$.

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The girth and dichromatic number

[Bokal et al, 2004] There are digraphs with arbitrary large digirth and arbitrary large finite dichromatic number.

How about uncountable dichromatic number?

[DS, 2016] Let $\lambda = \exp_n(\kappa)$ for some $2 \leq n < \omega$ and infinite κ. Then there is an orientation D of $\mathrm{Sh}_n(\lambda)$ so that whenever $G : [\lambda]^n \to \kappa$ then there is a monochromatic directed 4-cycle in D.

In particular, short odd cycles can be avoided while the dichromatic number is as large as we wish.

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Recall: $C_4 \hookrightarrow G$ if $\chi(G) > \omega$.

[DS, 2016] Consistently, for each $n \in \omega$ there is a digraph $D = D_n$ on vertex set ω_1 so that

1. D has no directed cycles of length $\leq n$, and
2. $\vec{C}_{n+1} \hookrightarrow D[X]$ for every uncountable $X \subseteq \omega_1$.

Consistently, there are graphs with uncountable dichromatic number and arbitrarily large digirth. Compactess arguments give the [Bokal et al, 2004] result as a corollary.

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Recall: large dichromatic # implies large chromatic # for the underlying graph.

[Erdős, Neumann-Lara, 1979] Is there a function $f : \mathbb{N} \to \mathbb{N}$ so that $\chi(G) \geq f(n)$ implies $\vec{\chi}(D) \geq n$ for some orientation D of G?

Even the existence of $f(3)$ is open.

Does $\chi(G) > \omega$ imply that $\vec{\chi}(D) > \omega$ for some **orientation D of G**?

Yes and no.

[DS, 2016] $\diamondsuit^+$ implies that **every graph** G **with** $\chi(G) = |G| = \omega_1$ has an orientation D so that $\vec{C}_4 \hookrightarrow D[X]$ whenever $\chi(G[X]) = \omega_1$.

- $\vec{C}_4$ can be substituted by **any finite bipartite** H.

[DS, 2016] **Consistently**, there is a graph G with $\chi(G) = |G| = \omega_1$ so that $\vec{\chi}(D) \leq \omega$ for any **orientation** D **of** G.

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A few open problems

Is there an elementary way of defining a digraph D with $\vec{\chi}(D) > \omega$?

Does $\vec{\chi}(D) > \omega$ imply that cycles of all but finitely many length embed into D?

Does $\vec{\chi}(D) > \omega$ imply that there is a strongly 2-connected subgraph of D?

Suppose that G has orientations D_ξ so that $\sup \vec{\chi}(D_\xi) = \kappa$. Is there a single orientation D with $\vec{\chi}(D) = \kappa$?

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