How to make infinite combinatorics simple?

Dániel T. Soukup

http://www.logic.univie.ac.at/∼soukupd73/

Introduction

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enumerate the objectives $\longrightarrow$ inductively meet these goals.

Colour the points of a topological space X with red and blue so that **both colors** appear on **any copy of the Cantor-space** in X.

- list all Cantor subspaces of X, and
- inductively declare one point red and one point blue from each.

If there are **more than $\mathfrak{c}$ such subspaces** then, after continuum many steps, we could have accidentally covered some Cantor-subspace with red points only.

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Goals - a short tutorial

- explore a general framework for inductive constructions,
  - arbitrary large structures by countable/continuum sized pieces,
  - how avoid the previous types of problems?
- demonstrate the applicability through entertaining examples:
  - paradoxical decompositions of the plane, and
  - Bernstein-decompositions of arbitrary topological spaces.

Based on

"Infinite combinatorics plain and simple" [ArXiv: 1705.06195]

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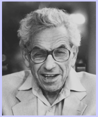

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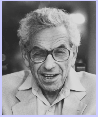

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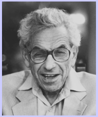

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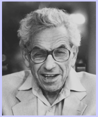

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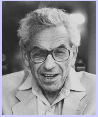

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CH implies that $\mathbb{R}^2 = S_0 \cup S_1 \cup \dots$ so that $|L \cap S_i| \leq 1$ for all $L \in \mathcal{L}_i$.

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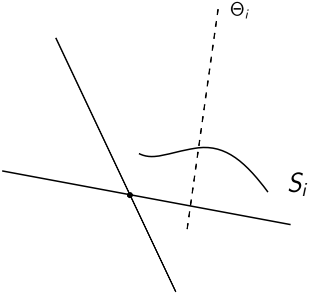

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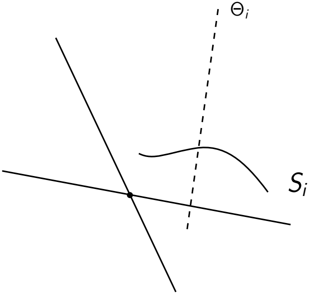

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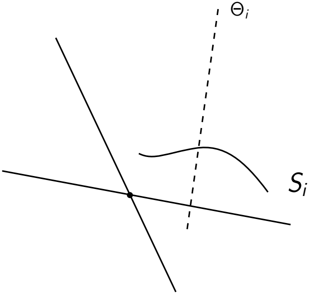

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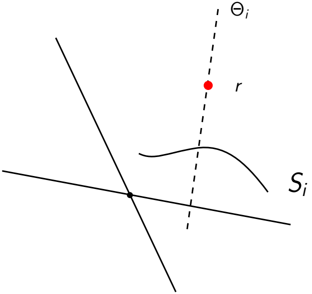

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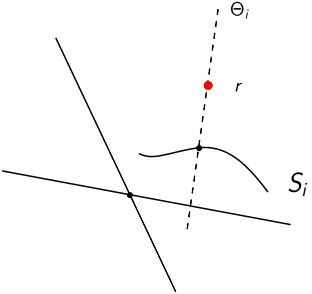

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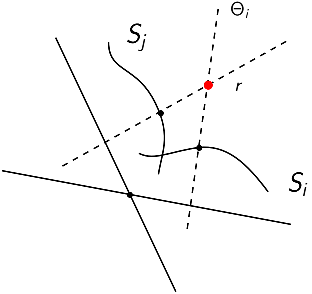

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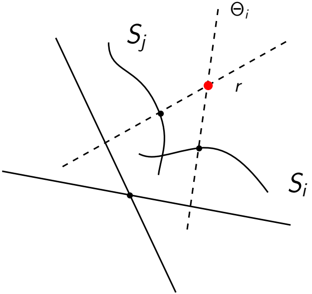

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CH implies that $\mathbb{R}^2$ is the union of countably many rotated graphs of functions.

If the union of S_i's is closed under constructibility then any new $r \in \mathbb{R}^2$ can be added to all but at most one S_i.

- use CH, to write $\mathbb{R}^2$ **as a continuous, increasing union of countable** R_α for $\alpha < \omega_1$,
- make sure that each R_α **is closed under constructibility**.
- if the S_i's union is R_α then list $R_{\alpha+1} \setminus R_\alpha$ as $\{t_n : n < \omega\}$,
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CH implies that $\mathbb{R}^2$ is the union of countably many rotated graphs of functions.

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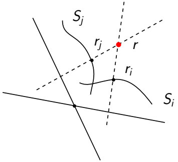

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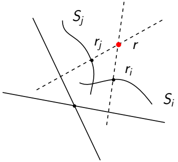

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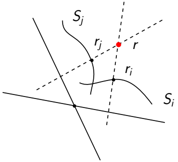

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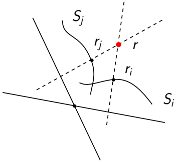

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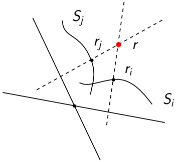

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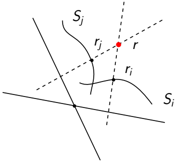

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The sequence $\langle R_\alpha \rangle_{\alpha<\omega_1}$ is called a filtration of $\mathbb{R}^2$ sometimes.

- any countable set is included in a countable set closed under constructibility;
- we could have closed under all first order operations, still countable;
- $H = H(\theta)$ and the real universe V agrees on properties of structures of size $<< \theta$;
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We take closure using elementary submodels: $M \prec H$ iff $M \subseteq H$ and

$$M \models \Phi \leftrightarrow H \models \Phi$$

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Given a set $R \subseteq H(\theta)$ of size $\aleph_1$ there is a continuous, increasing sequence of countable elementary submodels $\langle M_\alpha \rangle_{\alpha<\omega_1}$ of $H(\theta)$ covering R.

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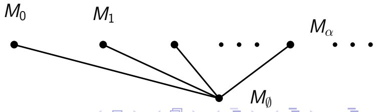

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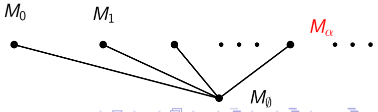

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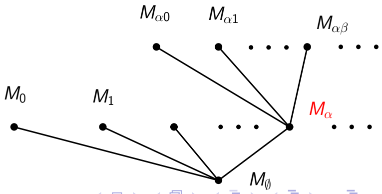

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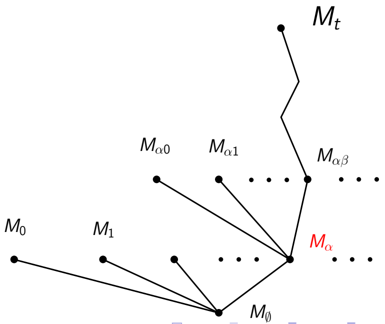

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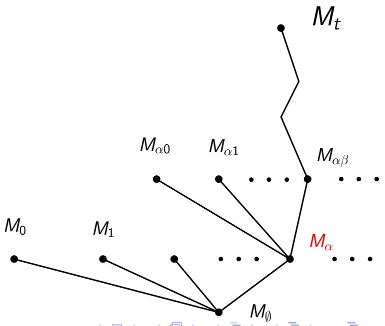

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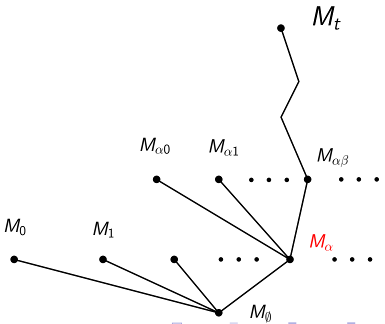

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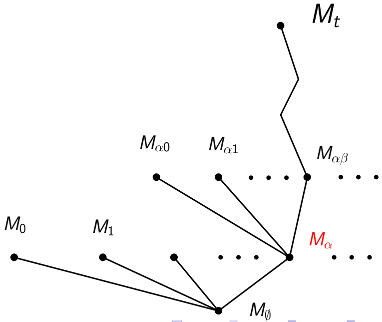

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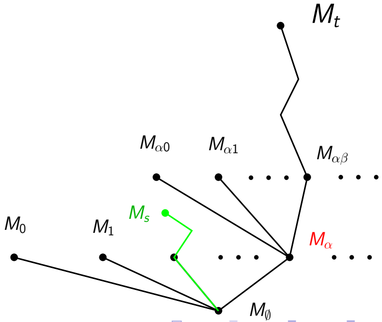

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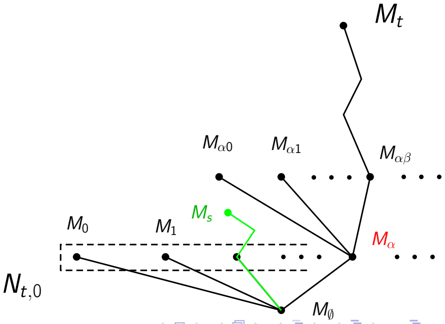

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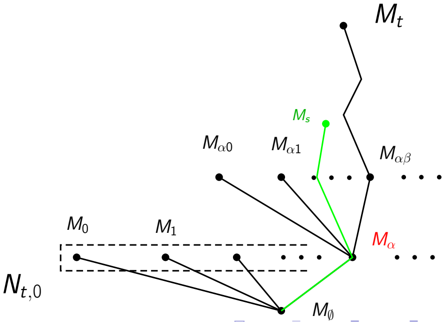

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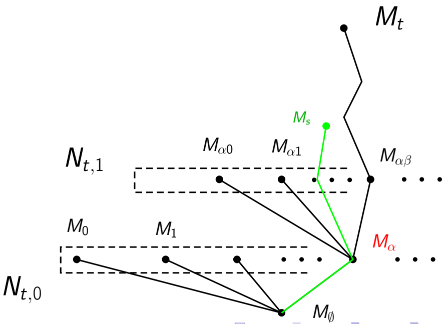

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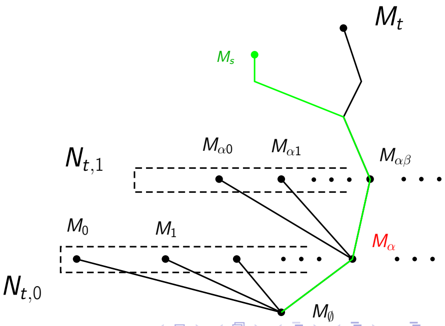

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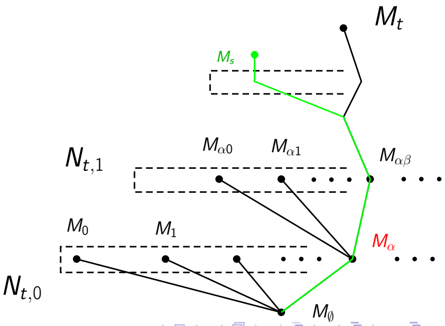

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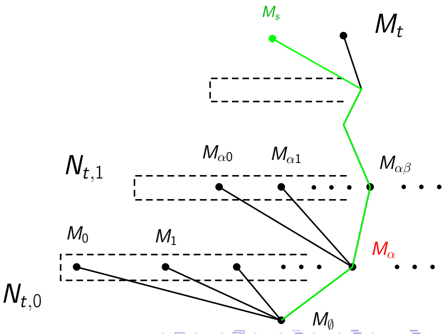

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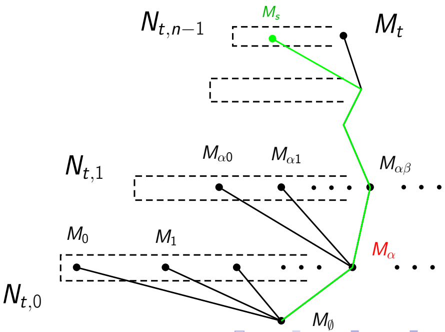

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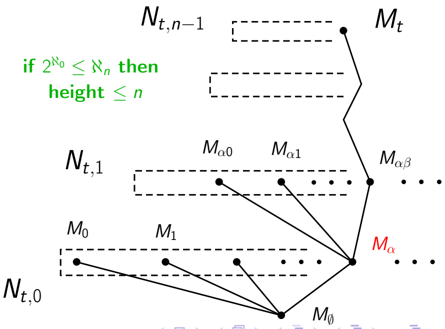

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$\mathbb{R}^2 = S_0 \cup S_1 \cup \dots$ so that $|L \cap S_i| \leq 1$ for all $L \in \mathcal{L}_i$.

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- so almost all choices of i_n works. □

Sierpinski without CH

$\mathbb{R}^2 = S_0 \cup S_1 \cup \dots$ so that $|L \cap S_i| \leq 1$ for all $L \in \mathcal{L}_i$.

Let $\langle M_\alpha \rangle_{\alpha<\mathfrak{c}}$ be a Davies-tree covering $\mathbb{R}^2$ so that $\Theta_i \in M_\alpha$.

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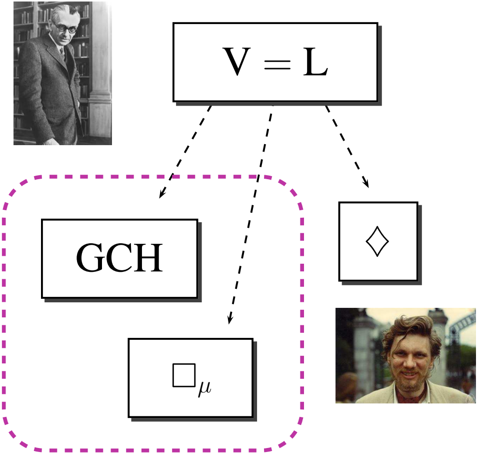

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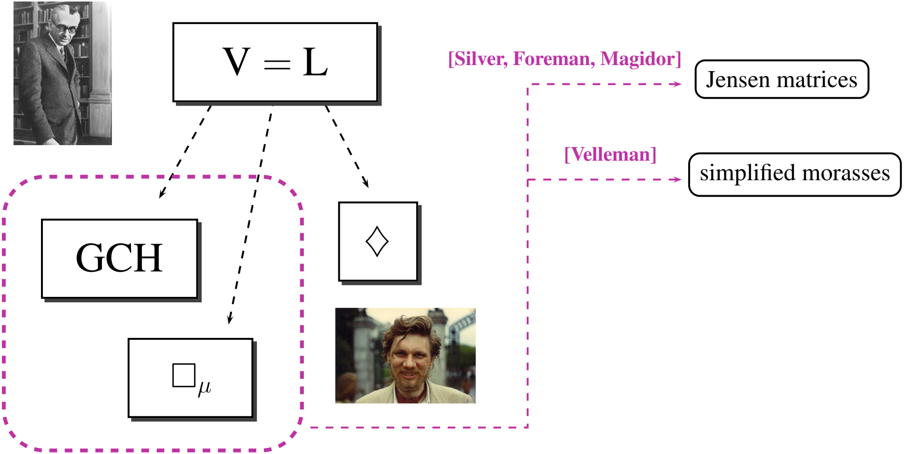

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[Velleman] simplified morasses

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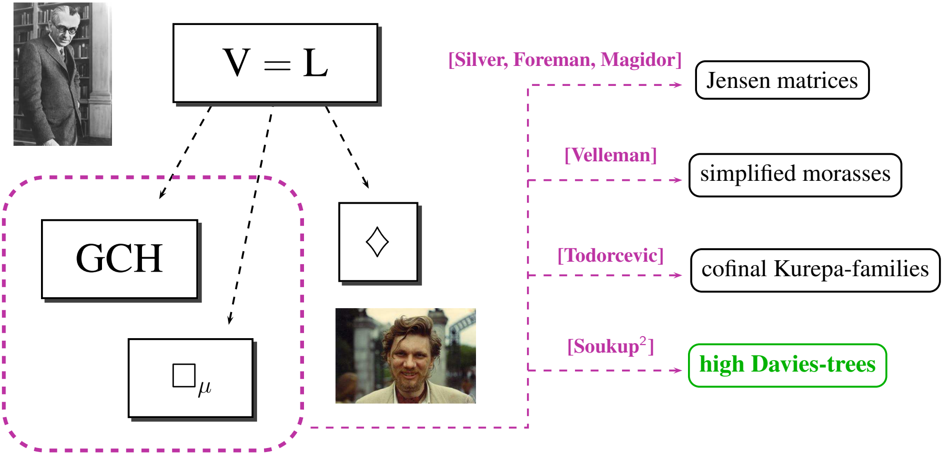

Countably closed models

Countable models $\to$ enumeration in type ω,
$\to$ deal with finite pieces one at a time.

M is countably closed if $x \subseteq M$, $|x| \leq \omega$ implies $x \in M$.

- for any $x \subseteq H(\theta)$ there is a countably closed $M \prec H(\theta)$ of size $|x|^\omega$;
- **c.c. models of size $\mathfrak{c}$** are very useful in various situations:
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Can we make Davies-trees from countably closed models of size $\mathfrak{c}$?

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High Davies-trees

We say that a **high Davies-tree for κ over x** is a sequence $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $H(\theta)$ for some large enough regular θ such that

(I) $[M_\alpha]^\omega \subset M_\alpha$, $|M_\alpha| = \mathfrak{c}$ and $x \in M_\alpha$ for all $\alpha < \kappa$,

(II) $[\kappa]^\omega \subset \bigcup_{\alpha<\kappa} M_\alpha$, and

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Note that $\kappa^\omega = \kappa$ if there is a high Davies-tree for κ.

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Theorem [DS, LS]

There are high Davies-tree **for any uncountable κ if $V = L$**.

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Main Theorem [DS, LS]

There are high Davies-tree for κ if $\kappa^\omega = \kappa$ and

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for all $\mathfrak{c} < \mu < \kappa$ with $\mathrm{cf}(\mu) = \omega$.

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for all $\mathfrak{c} < \mu < \kappa$ with $\mathrm{cf}(\mu) = \omega$.

Remark: **no high Davies-trees for $\kappa \geq \aleph_\omega$** if $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

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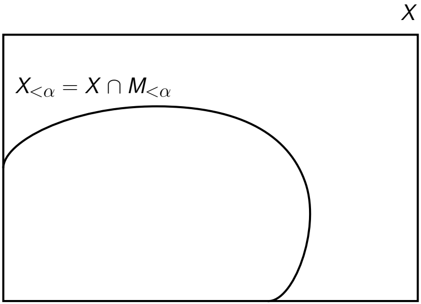

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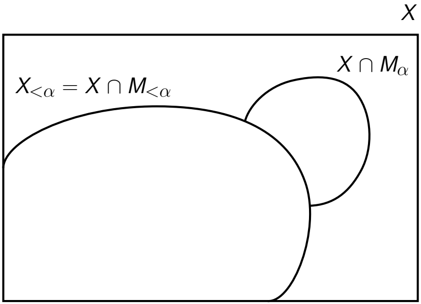

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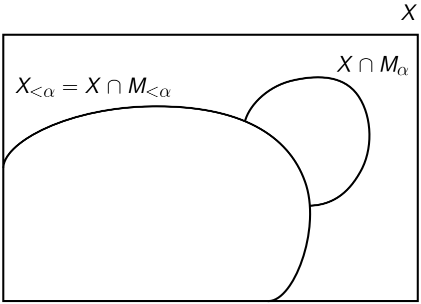

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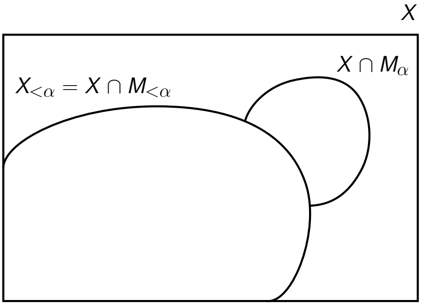

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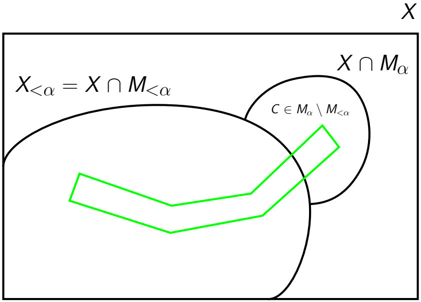

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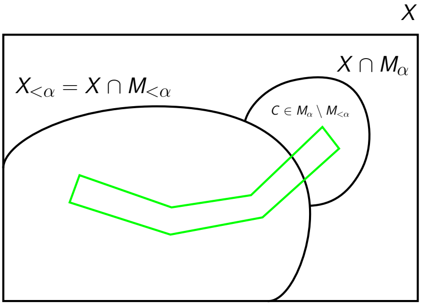

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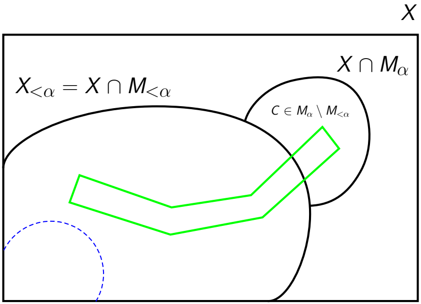

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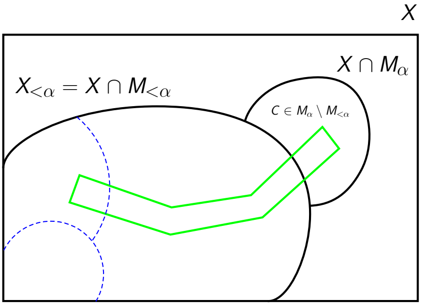

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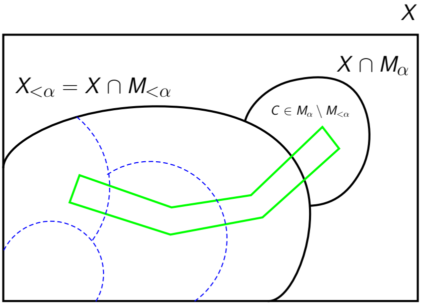

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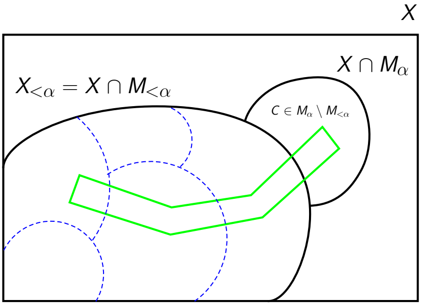

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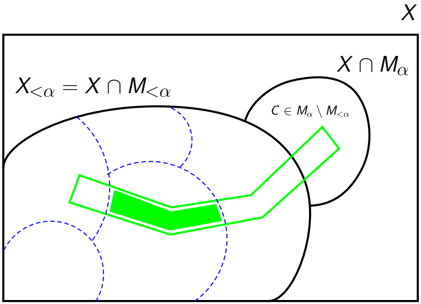

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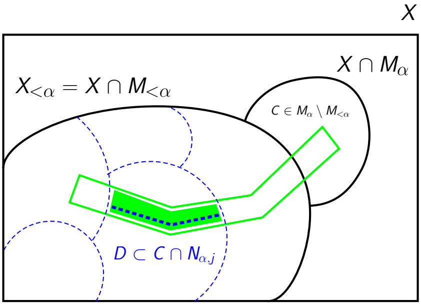

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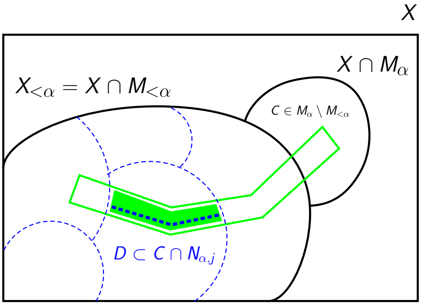

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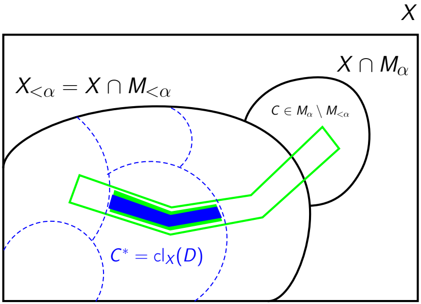

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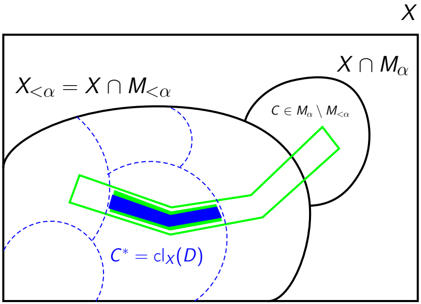

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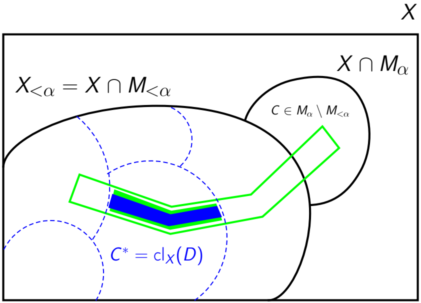

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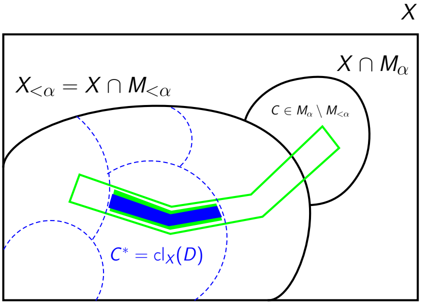

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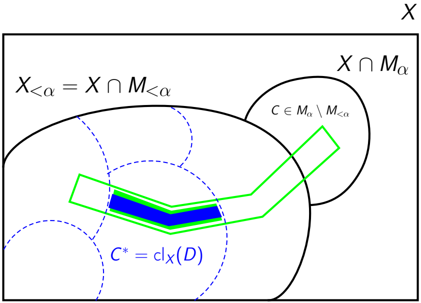

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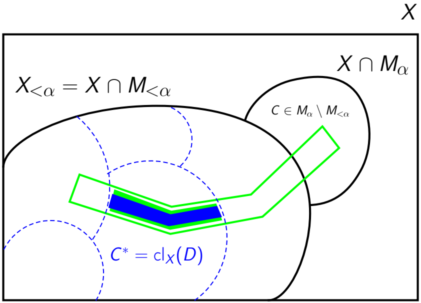

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Further applications

See "Infinite combinatorics plain and simple" at [ArXiv: 1705.06195] for more.

If $\exists$ **high Davies-trees for κ + CH** holds then

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Some open problems in the neighbourhood

There is $f : \mathbb{R}^n \to \omega$ such that there are

- no monochromatic rational distances [Komjáth], or
- no monochromatic triangles with non-zero rational area [Schmerl].

Folklore: there are $\mathfrak{c}$ points in the **Hilbert-space** ℓ^2 so that any two distinct points have rational distance.

[Komjáth] Are there $\mathfrak{c}$ points in ℓ^2 so that any three form a triangle with non-zero rational area?

A 2-point set $A \subseteq \mathbb{R}^2$ is such that $|A \cap \ell| = 2$ for every line $\ell \subset \mathbb{R}^2$.

[Sierpinski/Erdős] Is there a Borel 2-point set?

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- no monochromatic triangles with non-zero rational area [Schmerl].

Folklore: there are $\mathfrak{c}$ points in the **Hilbert-space** ℓ^2 so that any two distinct points have rational distance.

[Komjáth] Are there $\mathfrak{c}$ points in ℓ^2 so that any three form a triangle with non-zero rational area?

A 2-point set $A \subseteq \mathbb{R}^2$ is such that $|A \cap \ell| = 2$ for every line $\ell \subset \mathbb{R}^2$.

[Sierpinski/Erdős] Is there a Borel 2-point set?

Fremlin is offering £34 for "communicating a solution to him".
Efimov's problem pays £13, or £10 under MA + $\mathfrak{c} > \aleph_1$.