Enrichments of graphs with uncountable chromatic number

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Visit: www.logic.univie.ac.at/~soukupd73/
Papers, preprints and 'Combinatorial Set Theory' lecture notes.

What makes combinatorics interesting?

- Why (infinite) combinatorics?
  - Accessibility and diversity.
  - "A clever argument is beautiful to the problem-solver, a curiosity to a structuralist. [...] **It is the brilliant proofs, those that expand and/or transcend known technologies, which express the soul of the subject.**"

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  - "Combinatorics is the slums of topology." H. Whitehead
- Where does interesting combinatorics come from?
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Outline

- Intro to graphs and chromatic numbers;
- Review of partition relations and the arrow notation;
- 2-dimensional relations: orientations and edge-colourings;
- Higher dimensions;
- Classical open problems.

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Robarts Library

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Definition

The ***chromatic number*** *of a graph G, denoted by $\chi(G)$, is the least cardinal κ such that* ***the vertices of G can be covered by κ many independent sets****.*

Theme: **large chromatic number** versus **local sparsity**.

Boosting/ramifying partition relations.

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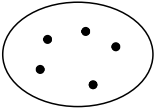

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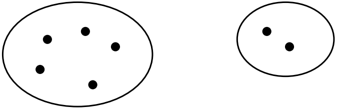

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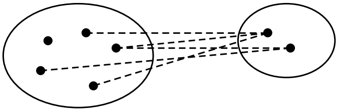

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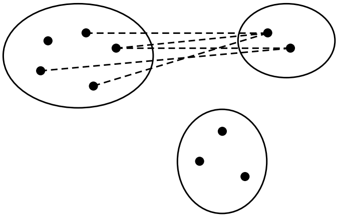

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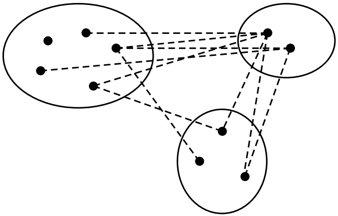

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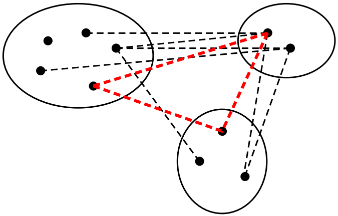

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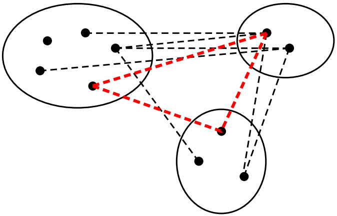

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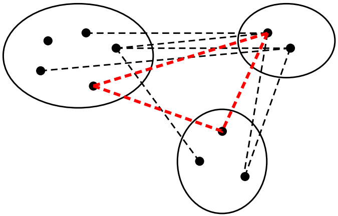

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Partition calculus

- Pigeonhole (dimension 1)
  For any $c : \mathbb{N} \to r$ with r finite, there is an infinite $A \subset \mathbb{N}$ so that $c \restriction A$ is constant.
  $$\mathbb{N} \to (\aleph_0)^1_r$$

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- 1-dimensional graph arrow (coloring vertices)
  For every colouring $c : V(G) \to \omega$, there is a monochromatic copy of H.
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Note: $\chi(G) > \omega$ if and only if $G \to (\text{edge})^1_\omega$.

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  - $G \to (\text{edge})^1_\omega$ implies $G \to (C_{2n})^1_\omega$ for any $n \geq 2$.
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- How fast?? [Erdős, Hajnal, and Szemerédi, 1982]
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Quiz 2: name the iconic neighbourhood.

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Kensington Market

The dichromatic number

- A digraph D is a pair (V, A) with $A \subset V^2$.
- An orientation of a graph $G = (V, E)$ is some digraph $D = (V, A)$ so that for any $\{u, v\} \in E$ either (u, v) or $(v, u) \in A$ (not both).

Ordered vertex set: for each edge, we decide if forward or backward.

The dichromatic number of a digraph D, denoted by $\vec{\chi}(D)$, is the least cardinal κ such that the vertices of D can be covered by κ many acyclic sets.

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Construction by Attila Joó, 2019:

- vertices are $V = n^\kappa$,
- $uv \in A$ iff

$$v(\delta) \equiv u(\delta) + 1 \mod n$$

for $\delta = \Delta(u, v)$.

- No cycles of length $< n$ but dichrom. $\geq \kappa$.

[DS, 2018] Consistently, for each $n \in \omega$ there is a digraph $D = D_n$ on vertex set ω_1 so that

- D has no directed cycles of length $\leq n$, and
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Size $\aleph_1$ in ZFC??

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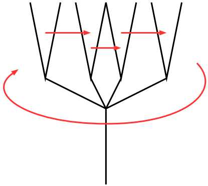

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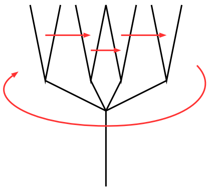

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How to get uncountable dichromatic number?

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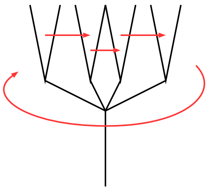

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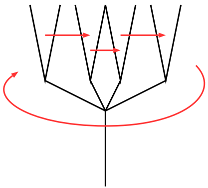

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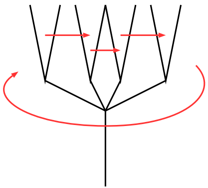

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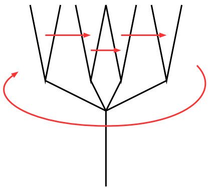

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Note: large dichromatic # implies large chromatic # for the underlying graph.

[Erdős, Neumann-Lara, 1979] Is there a function $f : \mathbb{N} \to \mathbb{N}$ so that $\chi(G) \geq f(n)$ implies $\overrightarrow{\chi}(D) \geq n$ for some orientation D of G?

Even the existence of $f(3)$ is open.

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Note: an orientation with $\vec{\chi}(D) > \omega$ is like an edge 2-colouring.

Simultaneous chromatic number

Let's say $\chi_r(G) > \omega$ if there is some edge r-colouring of G so that for any ω-partition of the vertices, one class has all the colours.

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Idea: Force $\chi(G) = \aleph_1$ then destroy witnesses to $\chi_2(G) > \omega$.

Consistently, there is a graph G so that

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Approach: ladder system graph and weak uniformization.

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Quiz 3: name the iconic set theorist.

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Paul Szeptycki

Looking forward - high dimensional relations

If $G \to (H)^1_\omega$ then the hypergraph $\binom{G}{H}$ has uncountable chromatic number. What can we say about this hypergraph?

- $H =$ a finite obligatory subgraph such as copies of C_4 or $K_{n,n}$;
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- Define anti-Ramsey hyper-edge-colourings!?

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Looking forward - high dimensional relations

If $G \to (H)^1_\omega$ then the hypergraph $\binom{G}{H}$ has uncountable chromatic number. What can we say about this hypergraph?

- $H =$ a finite obligatory subgraph such as copies of C_4 or $K_{n,n}$;
- $H =$ an infinite obligatory subgraph such as rays P_ω or half-graphs $H_{\omega,\omega}$.
- Define **anti-Ramsey hyper-edge-colourings!?**

Todorcevic, 1985 $P \to (\omega)^1_\omega$ implies $P \to (\alpha)^2_r$ for $r < \omega, \alpha < \omega_1$.

$$G \to (\omega)^1_\omega \text{ implies } G \to (H)^2_r \text{ for some 'large' } H??$$

Looking back - classical problems from Erdős

- Does $\chi(G) > \omega$ imply that there is a **Δ-free $H \hookrightarrow G$ with $\chi(H) > \omega$**?
- Does $G \to (K_3)^2_\omega$ imply $K_4 \hookrightarrow G$?

  [Consistently, no.]
- Does every two graphs G_0, G_1 with uncountable chromatic number contain **a common 4-chromatic subgraph?** Is there a common ω-chromatic subgraph?

Be inspired: Komjáth, P. "Erdős's Work on Infinite Graphs." Erdős Centennial. Springer, Berlin, Heidelberg, 2013. 325-345.

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Thank you very much! Questions?

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