Orientations of graphs with uncountable chromatic number

Dániel T. Soukup

http://www.logic.univie.ac.at/∼soukupd73/

Introduction

Goal: to study the chromatic number of uncountable digraphs.

- first organized effort on undirected case: P. Erdős and A. Hajnal in the 1960s;
- significant contributions: P. Komjáth, S. Shelah, C. Thomassen, S. Todorcevic...
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*The **chromatic number** of a graph G, denoted by $\chi(G)$, is the least cardinal κ such that **the vertices of G can be covered by κ many independent sets**.*

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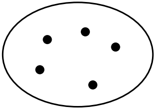

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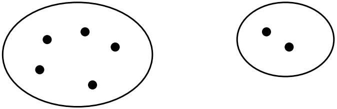

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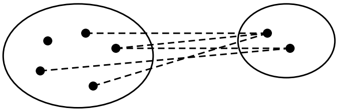

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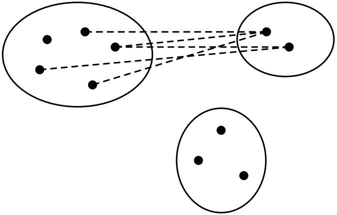

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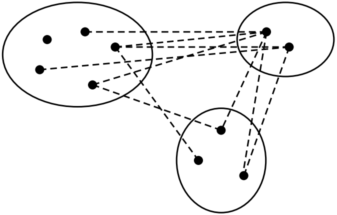

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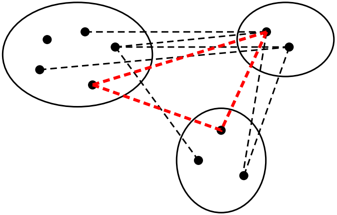

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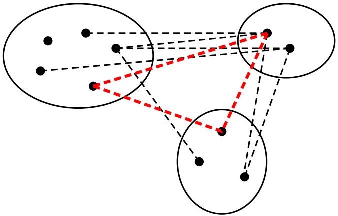

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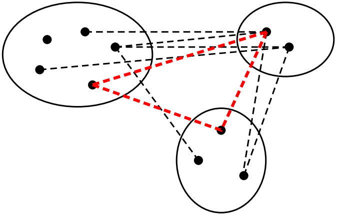

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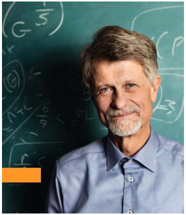

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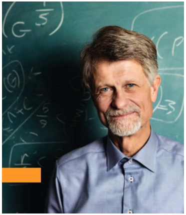

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Define the **shift graph** $\mathrm{Sh}_n(\lambda)$ (for $2 \leq n < \omega$) on $[\lambda]^n$ by connecting

$$u = \{\xi_0 < \cdots < \xi_{n-1}\} \text{ with } v = \{\xi_1 < \cdots < \xi_n\}.$$

If $\lambda = \exp_{n-1}(\kappa)^+$ then $\mathrm{Sh}_n(\lambda) \to (K_\kappa)^1_\kappa$. So $\chi(\mathrm{Sh}_n(\lambda)) > \kappa$.

No odd cycles of length $\leq 2n - 1$.

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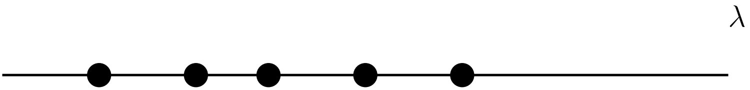

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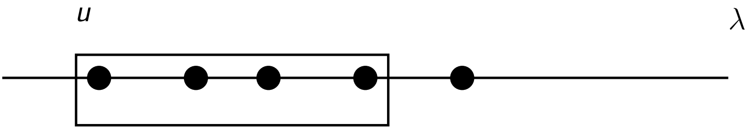

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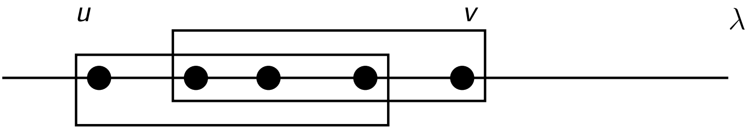

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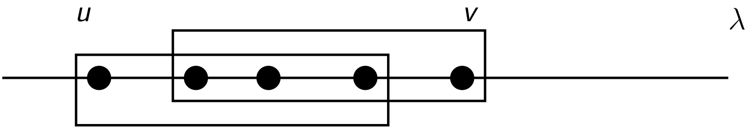

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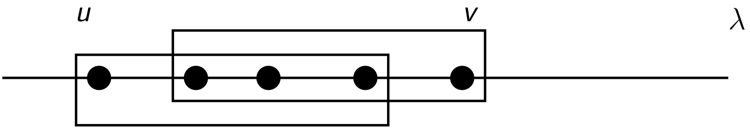

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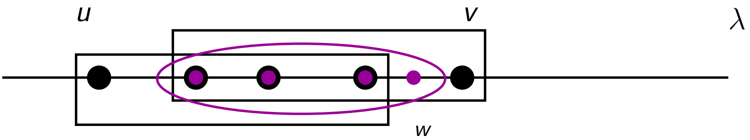

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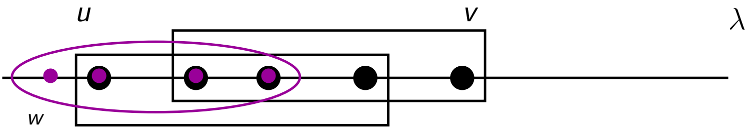

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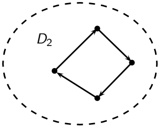

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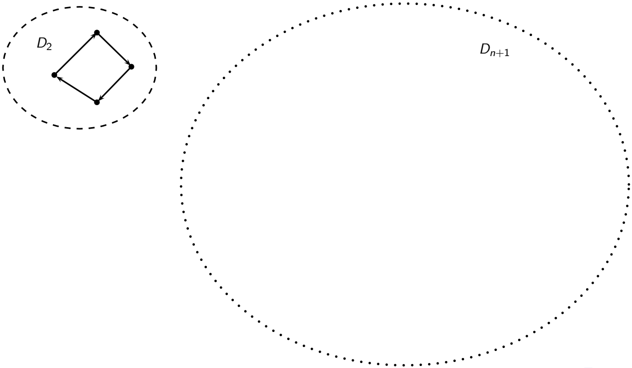

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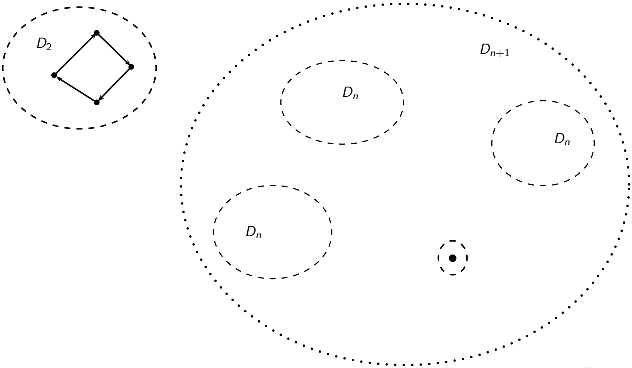

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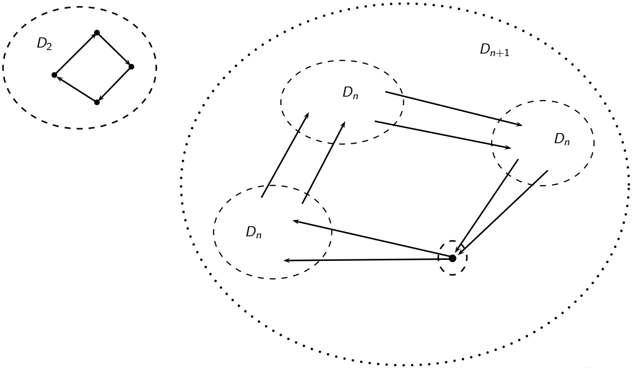

Uncountable tournaments

How to find D with $\vec{\chi}(D) > \omega$?

[J. Moore] Let $f : [\omega_1]^2 \to 2$ so that if $A, B \subseteq \omega_1$ are uncountable and $i < 2$ then $f(\alpha, \beta) = i$ for some $\alpha \in A, \beta \in B$ with $\alpha < \beta$.
Let $\alpha\beta \in E$ **iff** $\alpha < \beta$ **and** $f(\alpha, \beta) = 0$; otherwise $\beta\alpha \in E$.

If $X \subseteq \omega_1$ is uncountable then $\vec{C}_3 \hookrightarrow D[X]$.

- Find $u \in X$ so that $A = N^+(u) \cap X$ and $B = N^-(u) \cap X$ are uncountable.
- Now pick $\alpha < \beta$ so that $\alpha \in A, \beta \in B$ and $f(\alpha, \beta) = 0$. Then $\{u, \alpha, \beta\} = \vec{C}_3$ in $D[X]$.

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Alternate proof: **diagonalization of length $\mathfrak{c}$** using countable elementary submodels.

Uncountable tournaments

How to find D with $\vec{\chi}(D) > \omega$?

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Recall: there are digraphs with large girth and large finite dichromatic number.

How about **uncountable dichromatic number?**

[DS, 2016] Let $\lambda = \exp_n(\kappa)$ for some $2 \leq n < \omega$ and infinite κ. Then **there is an orientation** D **of** $\mathrm{Sh}_n(\lambda)$ so that whenever $G : [\lambda]^n \to \kappa$ then **there is a monochromatic directed 4-cycle in D**.

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- $D \to (D_0)^1_r$ iff for every r-colouring of the vertices of D one can find a monochromatic copy of D_0.
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Recall: $C_4 \hookrightarrow G$ if $\chi(G) > \omega$.

[DS, 2016] Consistently, for each $n \in \omega$ there is a digraph $D = D_n$ on vertex set ω_1 so that

1. D has no directed cycles of length $\leq n$, and
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Fix $n \in \omega$.

- Let $p \in \mathbb{P} = \mathbb{P}_n$ iff p is a finite digraph of girth $> n$ on a subset of ω_1.
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Goal: $\mathbb{P}$ is ccc.

Let $\dot{G} \subseteq \mathbb{P}$ generic filter. Define $\dot{D} = (\omega_1, E^G)$ by $E^G = \bigcup\{E^p : p \in G\}$.

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Suppose that $\{D_i : i \leq k\}$ **are isomorphic digraphs of girth** $> n$ forming a Δ-system with isomorphism $\psi_{ij} : V_i \to V_j$. Let $D = \bigcup\{D_i : i \leq k\}$.

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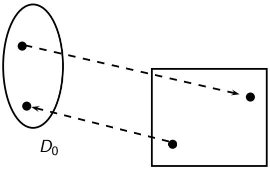

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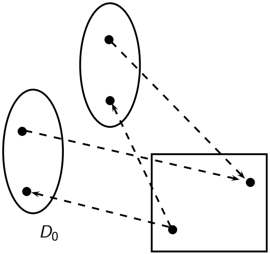

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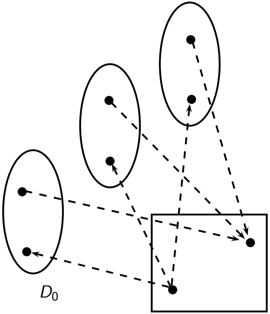

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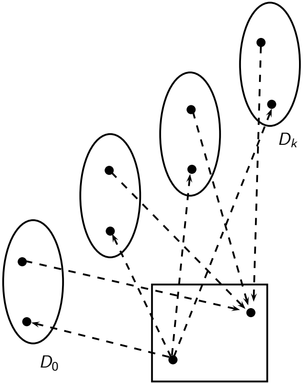

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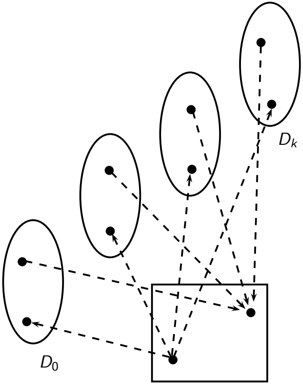

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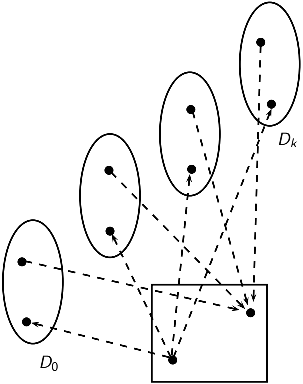

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Suppose that $\{D_i : i \leq k\}$ **are isomorphic digraphs of girth** $> n$ forming a Δ-system with isomorphism $\psi_{ij} : V_i \to V_j$. Let $D = \bigcup\{D_i : i \leq k\}$.

- Any path P from $\alpha \in V_i$ to $\alpha' = \psi_{ij}(\alpha) \in V_j$ in D has length $> n$.
- D has girth $> n$.
- If $k \geq n$ and

$$E(D^*) = E(D) \cup \{\alpha_k\alpha_0, \alpha_i\alpha_{i+1} : i < k\}$$

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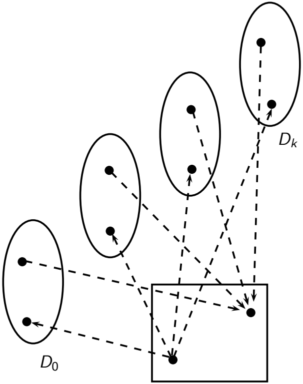

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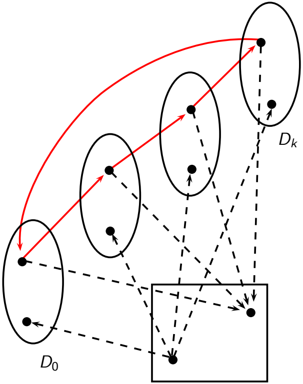

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Suppose that $\chi(D) > \omega$ for some digraph D. Does

$$D \to (\vec{C}_n)^1_\omega$$

hold for **some** $n = n(D)$**?** Yes, for all the previous examples...

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Recall: $\overrightarrow{\chi}(G) = \sup\{\overrightarrow{\chi}(D) : D$ is an orientation of $G\}$

What happens if $\overrightarrow{\chi}(G)$ is a limit cardinal?

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Recall: we can consistently avoid finitely many cycles.

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D is **strongly n-connected** iff for any vertices u, v and finite set F of size $< n$ there is a directed path from u to v avoiding vertices in F.

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- If $\chi(G) > \omega$ and $n \in \omega$ then $K_{n,n} \hookrightarrow G$ is an n-connected subgraph.
- **[Komjáth]** If $\chi(G) > \omega$ and $n \in \omega$ the $\chi(G_0) > \omega$ for some n-connected $G_0 \hookrightarrow G$.

Does $\vec{\chi}(D) > \omega$ imply that $D_0 \hookrightarrow D$ for some **strongly 2-connected digraph** D_0?

There could be no uniform witness: the girth of D can be arbitrary large.

Open problems - connected subgraphs

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Any questions?