HITCHHIKER'S GUIDE TO COLORING PAIRS OF $\aleph_2$

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Abstract. The aim of this note is to present Shelah's celebrated coloring theorem [4] of pairs of ω_2 by closely following S. Todorcevic [7]. The only aim of this paper is to collect all preliminary results and standard tricks in a single place which hopefully makes the results more accessible to interested non-specialists.

1. Introduction

Our goal is to present a proof of the following theorem in a self-contained manner:

Main Theorem. *There is a $c : [\omega_2]^2 \to \omega_2$ such that for every $n \in \omega$, for every pairwise disjoint $A \in [[\omega_2]^n]^{\omega_2}$ and every $h : n \times n \to \omega_2$ there is $a < b \in A$ such that*

$$c(a(i), b(j)) = h(i, j)$$

for all $i, j \in n$.

Here $a < b$ means that $\alpha < \beta$ for all $\alpha \in a, \beta \in b$ and $a(i)$ denotes the ith element of a with respect the natural ordering.

We will prove the Main Theorem in four stages as follows:

- Stage I: we define a standard "square bracket" coloring f on ω_1,
- Stage II: using the above coloring f, we define a map $g : \omega_1^{<\omega} \to \omega_1$ with strong properties reminiscent of square bracket colorings[1],
- Stage III: we construct a map $c_0 : [\omega_2]^2 \to \omega_1$ which has the universal properties of our Main Theorem with ω_1-many colors and constant functions h[2],
- Stage IV: we modify c_0 in two smaller, independent steps to get a coloring $c : [\omega_2]^2 \to \omega_2$ satisfying the requirements of the Main Theorem.

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[1] Pr_6-type coloring in Shelah's notation

[2] Pr_1-type colorings in Shelah's notation

In the proofs, the reader will find the definition of the corresponding coloring in the first paragraph; this makes it possible to get a sense of the idea while not spending much time checking the details.

Finally, it is not our goal to show the importance of the Main Theorem however we mention two application.

Shelah's original motivation was to prove that there are two $\aleph_2$-chain condition posets P_0, P_1 (i.e. no anti-chains of size $\aleph_2$ in P_i) with the product containing an anti-chain of size $\aleph_2$. Recall that the same problem for $\aleph_1$-chain condition is undecidable in ZFC [8]. Define

$$P_i = \{p \in [\omega_2]^\omega : c``[p]^2 = \{i\}\}$$

with c from the Main Theorem and $i \in 2$. Then P_0, P_1 witnesses the above fact.

The second application is to topology; a significant portion of set theoretic topology in the 20th century revolved around the problem of S-and L-spaces [3], i.e. determining the connection between regular hereditarily separable (HS) and hereditarily Lindelöf spaces (HL). It was soon realized that there are many models of ZFC where these properties do not imply each other i.e. the existence of S-spaces and L-spaces. The research culminated in two deep results: S. Todorcevic proved that PFA implies that every HS space must be HL [5] (nicely worked out at [2]) and J. Moore proved that there is, in ZFC, an HL space which is not HS [1].

Naturally, one can ask the same question for higher cardinals thus defining the class of κ-HL and κ-HS spaces. The coloring of Main Theorem provides us ω_2-HL spaces which are not ω_2-HS and vica versa:

Theorem 1.1. *There are dense subsets $X, Y \subseteq 2^{\omega_2}$ such that X is right separated in order type ω_2 and has density ω_1 while Y is left separated in order type ω_2 and has Lindelöf-degree ω_1.*

It is an intriguing open problem whether there are compact spaces with the above properties.

Stage I

Our first theorem is the now classical result of S. Todorcevic [6]:

Theorem 1.2. *There is $f : [\omega_1]^2 \to \omega_1$ such that for every $n \in \omega$ and $A = \{\tau_\gamma : \gamma \in \omega_1\} \subseteq \omega_1^n$ with pairwise disjoint range and every $\xi \in \omega_1$ there is $\zeta < \gamma < \omega_1$ such that*

$$f(\tau_\zeta(i), \tau_\gamma(i)) = \xi$$

for all $i < n$.

We will (locally) refer to the above situation as *realizing the color* ξ.

This theorem is usually stated for pairwise disjoint finite sets $\{\tau_\gamma : \gamma \in \omega_1\}$ rather than finite sequences with disjoint range. Also, the reader first encountering this argument is encouraged to work out the $n = 1$ case only, i.e. where A is simply an uncountable subset of ω_1.

Proof. Note that it suffices to construct $f_0 : [\omega_1]^2 \to \omega_1$ such that the set of $\xi < \omega_1$ realized by the coloring f_0 contains a club. Indeed, if the map $\xi \mapsto \xi^*$ takes each value stationary often then the coloring $\boxed{f(\alpha,\beta) = f_0(\alpha,\beta)^*}$ realizes all colors.

Now, pick a 1-1 sequence of reals $R = \{r_\alpha : \alpha < \omega_1\} \subseteq 2^\omega$ and let

$$\Delta(\alpha,\beta) = \min\{n \in \omega : r_\alpha(n) \neq r_\beta(n)\}$$

for $\alpha < \beta < \omega_1$. Also, fix a sequence $e_\alpha : \alpha \to \omega$ of 1-1 maps. Let

$$\boxed{f_0(\alpha,\beta) = \min\big((e_\beta^{-1}(\Delta(\alpha,\beta)) \cup \{\beta\}) \setminus \alpha\big)}$$

for $\alpha < \beta < \omega_1$.

We prove that f_0 works for club many colors $\xi \in \omega_1$; indeed, we prove that if $\xi = M \cap \omega_1$ for some countable elementary submodel of $H(\aleph_2)$ with $\omega_1, R, (e_\alpha)_{\alpha\in\omega_1} ... \in M$ then we can realize the color ξ.

Fix M as above and let $\xi = M \cap \omega_1$; pick any τ_γ such that $b = \operatorname{ran}\tau_\gamma \subseteq \omega_1 \setminus \xi$. Let $m = \max\{e_\beta(\xi) : \beta \in b\} + 1$ and consider the set

$$B = \{\zeta < \omega_1 : \forall i < n : r_{\tau_\zeta(i)} \restriction m = r_{\tau_\gamma(i)} \restriction m\}.$$

It should be clear that $B \in M$ and $\gamma \in B$ which implies that $|B| = \omega_1$. Let

$$\mathcal{S} = \{s \in (2^{<\omega})^n : \forall i < n : s_i = r_{\tau_\gamma(i)} \restriction (|s_i| - 1)^\frown(1 - r_{\tau_\gamma(i)}(|s_i|))\}$$

and let $B_s = \{\zeta \in B : (r_{\tau_\zeta(i)} \restriction |s_i|)_{i<n} = s\}$ for $s \in S$. Note that an $s \in \mathcal{S}$ simply defines the place where the sequence of reals $(r_{\tau_\zeta(i)})_{i<n}$ first differs from $(r_{\tau_\gamma(i)})_{i<n}$.

Then $B \setminus \{\gamma\} = \cup\{B_s : s \in \mathcal{S}\}$ thus there is $s \in \mathcal{S}$ such that $|B_s| = \omega_1$. Note that the value of $(\Delta(\tau_\zeta(i), \tau_\gamma(i)))_{i<n} \in (\omega \setminus m)^n$ is constant if ζ runs over B_s. Call this sequence $(m_i)_{i<n}$ and let

$$F = \cup\{e^{-1}_{\tau_\gamma(i)}(m_i) : i < n\}.$$

Note that $F \cap \xi$ is a finite set in M and $B_s \in M$ is uncountable thus we can pick $\zeta \in B_s$ such that $\operatorname{ran}\tau_\zeta \subseteq \omega_1 \setminus (\max F + 1)$. It is now straightforward to check that the above choices guarantee $f_0(\tau_\zeta(i), \tau_\gamma(i)) = \xi$ for all $i < n$. $\square$

Stage II

The next somewhat technical result will play a key role in defining the final colorings; although at this point, it is very far from trivial to see how we will utilize this particular map to define a coloring on pairs of ω_2.

Theorem 1.3. *There is $g : \omega_1^{<\omega} \to \omega_1$ such that for every $\{\tau_\gamma : \gamma \in \omega_1\} \subseteq \omega_1^{<\omega}$ with $\gamma \in \operatorname{ran}(\tau_\gamma)$ and every $\xi \in \omega_1$ there is $\gamma < \delta < \omega_1$ such that*

$$g(\sigma^\frown\vartheta) = \xi$$

for all $\sigma \subseteq \tau_\gamma, \vartheta \subseteq \tau_\delta$ with $\gamma \in \operatorname{ran}(\sigma)$ and $\delta \in \operatorname{ran}(\vartheta)$.

Proof. Pick a 1-1 sequence of reals $R = \{r_\alpha : \alpha < \omega_1\} \subseteq 2^\omega$ and let

$$\Delta(\tau) = \max\{\Delta(r_{\tau(i)}, r_{\tau(j)}) : i, j < n, \tau(i) \neq \tau(j)\}$$

for any non constant $\tau \in \omega_1^{<\omega}$, i.e. Δ finds the minimal distance appearing in the set $\{r_{\tau(i)} : i < n\}$. In particular, the reals $\{r_{\tau(i)} : i < |\tau|\}$ already differ restricted to $\Delta(\tau) + 1$. We fix a coloring f as defined in Stage I and let

$$\boxed{g(\tau) = f(\tau(i), \tau(j))}$$

where $(i, j) = \min\{(p, q) \in |\tau|^2 : \Delta(\tau) = \Delta(r_{\tau(p)}, r_{\tau(q)})\}$ and $\tau \in \omega_1^{<\omega}$ is non constant; otherwise $g(\tau) = 0$ for constant $\tau \in \omega_1^{<\omega}$.

We prove now that g works; pick a set $\{\tau_\gamma : \gamma \in \omega_1\} \subseteq \omega_1^{<\omega}$ as above and note that we can suppose that the sequences are non constant. Otherwise, τ_γ is constant γ and $g(\sigma^\frown\vartheta) = f(\gamma, \delta)$ for all $\sigma \subseteq \tau_\gamma, \vartheta \subseteq \tau_\delta$ with $\gamma \in \operatorname{ran}(\sigma)$ and $\delta \in \operatorname{ran}(\vartheta)$. Thus each color $\xi \in \omega_1$ is realized by the choice of f.

Now, we can thin out the sequence $\{\tau_\gamma : \gamma \in \omega_1\}$ so that there are $k, l, m \in \omega$ and $I \in [\omega_1]^{\omega_1}$:

(i) $|\tau_\gamma| = l$ and $\Delta(\tau_\gamma) = m$ for all $\gamma \in I$,
(ii) $(r_{\tau_\gamma(i)} \restriction m + 1)_{i<l}$ is constant in $\gamma \in I$,
(iii) $\{\operatorname{ran} \tau_\gamma : \gamma \in I\}$ is a Δ-system with root c,
(iv) $\tau_\gamma(k) = \gamma$ for all $\gamma \in I$ and
(v) $\Delta(r_{\tau_\gamma(i)}, r_{\tau_\delta(i)}) > m$ for all $\gamma < \delta \in I$ and $i < l$,
(vi) $\tau_\gamma(i) = \tau_\gamma(j)$ iff $\tau_\zeta(i) = \tau_\zeta(j)$ for all $\gamma < \zeta \in I$ and $i < l$.

Fix an arbitrary color $\xi \in \omega_1$ which we wish to realize; by the properties of f, we can find $\gamma < \delta \in I$ such that

$$f(\tau_\gamma(i), \tau_\delta(i)) = \xi$$

for all $i < l$ and $\tau_\gamma(i) \in \operatorname{ran} \tau_\gamma \setminus c$. We wish to show that γ, δ works for g so fix $\sigma \subseteq \tau_\gamma, \vartheta \subseteq \tau_\delta$ with $\gamma \in \operatorname{ran}(\sigma)$ and $\delta \in \operatorname{ran}(\vartheta)$. Note that γ

and δ share the same position in $\operatorname{ran}\tau_\gamma \setminus c$ and $\operatorname{ran}\tau_\delta \setminus c$ by (iv) thus $\Delta(\gamma,\delta) > m$ by (v). Hence $\Delta(\sigma^\frown\vartheta) > m$. Let $g(\sigma^\frown\vartheta) = f(\tau(i),\tau(j))$ where (i,j) is the pair picked by the definition of g. Let $\zeta = \sigma^\frown\vartheta$ and note that $\zeta(i)$ and $\zeta(j)$ cannot be both in $\operatorname{ran}\tau_\gamma$ or $\operatorname{ran}\tau_\delta$ by (i). Also, i and j must share the same position in the running part of the Δ-system; otherwise (ii) implies that $r_{\zeta(i)}$ and $r_{\zeta(j)}$ already differ below m which contradicts $\Delta(\zeta) = \Delta(r_{\zeta(i)}, r_{\zeta(j)}) > m$. Hence, there is $n < l$ such that $\tau_\gamma(n) = \zeta(i)$ and $\tau_\delta(n) = \zeta(j)$ (or vica versa $\tau_\gamma(n) = \zeta(j)$ and $\tau_\delta(n) = \zeta(i)$) and

$$g(\zeta) = f(\tau_\gamma(n), \tau_\delta(n)) = \xi.$$

□

Stage III

Before defining the coloring of this stage, we need some further notions.

Definition 1.4. *A* C-sequence on ω_2 *is a sequence* $\{C_\alpha : \alpha \in \omega_2\}$ *such that for all* $\alpha \in \omega_2$*:*

(1) C_α *is a closed unbounded subset of* α *with order type equal to the cofinality of* α*,*
(2) if $\zeta \in C_\alpha$ *is a successor element of* C_α *then* ζ *is a successor ordinal.*

We fix a C-sequence $\{C_\alpha : \alpha \in \omega_2\}$ for the rest of the section.

The *minimal walk* (along the C-sequence) from β to α (where $\alpha < \beta < \omega_2$) is a finite decreasing sequence $\beta = \beta_0, \beta_1, ..., \beta_n = \alpha$ defined recursively by

$$\beta_{i+1} = \min C_{\beta_i} \setminus \alpha$$

provided $\beta_i \neq \alpha$.

The set $\{\beta_0 ... \beta_n\}$ is called the *upper trace* of the walk and is denoted by $Tr(\alpha,\beta)$. Furthermore, let

$$\lambda(\alpha,\beta) = \max\{\max(C_{\beta_i} \cap \alpha) : \beta_i \in Tr(\alpha,\beta) \setminus \{\alpha\}\}$$

for $\alpha < \beta < \omega_2$. Note that $\lambda(\alpha,\beta) < \alpha$ and

Observation 1.5. *For every* $\alpha < \beta < \gamma < \omega_2$ *such that* $\lambda(\beta,\gamma) < \alpha$ *the minimal walk from* γ *to* β *is an initial segment of the walk from* γ *to* α*.*

We are now ready to define the coloring of this stage:

Theorem 1.6. *There is a* $c_0 : [\omega_2]^2 \to \omega_1$ *such that for every* $n \in \omega$, *for every pairwise disjoint* $A \in [[\omega_2]^n]^{\omega_2}$ *and every* $\xi < \omega_1$ *there is* $a < b \in A$ *such that*

$$c_0(a(i), b(j)) = \xi$$

for all $i, j \in n$.

Proof. Let $\delta \mapsto \delta^*$ map ω_2 to ω_1 such that

$$\Delta_\xi = \{\delta \in S^{\omega_2}_{\omega_1} : \delta^* = \xi\} \text{ is stationary}$$

for all $\xi \in \omega_1$ where $S^{\omega_2}_{\omega_1} = \{\delta \in \omega_2 : cf(\delta) = \omega_1\}$. Now compose the function $*$ with the trace function to obtain

$$\rho_* : [\omega_2]^2 \to \omega_1^{<\omega}$$

that is

$$\boxed{\rho_*(\alpha, \beta) = (\beta_i^*)_{i<n}}$$

where $\alpha < \beta < \omega_2$ and $(\beta_i^*)_{i<n}$ is the walk from β to α. Finally, consider the map $g : \omega_1^{<\omega} \to \omega_1$ constructed in Stage II and define $c_0 : [\omega_2]^2 \to \omega_1$ by

$$\boxed{c_0(\alpha, \beta) = g(\rho_*(\alpha, \beta))}$$

for $\alpha < \beta < \omega_2$.

We claim that c_0 works! Fix $n \in \omega$ and pairwise disjoint $A \in [[\omega_2]^n]^{\omega_2}$. We wish to set up a situation from $\aleph_1$-many elements of A so that the minimal walks between these element meet each Δ_ξ in a nice way; in particular, the sequences defined by ρ_* will contain every element of ω_1.

More precisely, let

$$C = \{\delta \in \omega_2 : \forall \gamma < \delta \exists a \in A : a \subset (\gamma, \delta)\}$$

and note that C is a club. Fix $b_\delta \in A$ such that $\delta < b_\delta$ for all $\delta \in \omega_2$.

Claim 1.7. *There is* $\lambda_0 \in \omega_2$ *and stationary* $\Sigma_\xi \subset \Delta_\xi \cap C$ *for* $\xi \in \omega_1$ *such that*

$$\lambda(\delta, \beta) < \lambda_0$$

for all $\xi \in \omega_1, \delta \in \Sigma_\xi$ *and* $\beta \in b_\delta$.

Proof. Recall that $\lambda(\delta, \beta) < \delta$ if $\delta \in S^{\omega_2}_{\omega_1}$ and $\delta < \beta < \omega_2$ and that $\Delta_\xi \cap C$ is stationary for each $\xi \in \omega_1$; thus the claim follows from the Pressing Down Lemma. □

Fix $\varepsilon \in S^{\omega_2}_{\omega_1} \cap \bigcap\{\lim \Sigma_\xi : \xi < \omega_1\} \setminus \lambda_0$ and $\delta_\xi \in \Sigma_\xi \setminus (\varepsilon+1)$ for $\xi \in \omega_1$. Pick a cofinal sequence $(\gamma_\xi)_{\xi<\omega_1}$ in ε such that $\gamma_\xi \in \Sigma_\xi$ and

$$\bigcup\{\lambda(\varepsilon, \beta) : \beta \in \cup_{\eta<\xi} b_{\delta_\eta}\} < \gamma_\xi$$

for all $\xi < \omega_1$.

Pick $a_\xi \in A$ such that

(i) $\sup\{\gamma_\eta : \eta < \xi\} < a_\xi < \gamma_\xi$,
(ii) $\lambda(\gamma_\xi, \varepsilon) < a_\xi$,
(iii) $\bigcup\{\lambda(\varepsilon, \beta) : \beta \in \cup_{\eta<\xi} b_{\delta_\eta}\} < a_\xi$

for $\xi < \omega_1$. We made the above choices in order to ensure that

$$\rho_*(\alpha, \varepsilon) = \rho_*(\gamma_\xi, \varepsilon)^\frown \rho_*(\alpha, \gamma_\xi), \tag{1.1}$$

$$\rho_*(\varepsilon, \beta) = \rho_*(\delta_\eta, \beta)^\frown \rho_*(\varepsilon, \delta_\eta), \tag{1.2}$$

and

$$\rho_*(\alpha, \beta) = \rho_*(\varepsilon, \beta)^\frown \rho_*(\alpha, \varepsilon) \tag{1.3}$$

if $\eta < \xi < \omega_1, \alpha \in a_\xi$ and $\beta \in b_{\delta_\eta}$. Most importantly, $\eta \in \operatorname{ran} \rho_*(\varepsilon, \beta)$ and $\xi \in \operatorname{ran} \rho_*(\alpha, \varepsilon)$!

Now, let

$$\tau_\xi = \big(\rho_*(\alpha, \varepsilon)\big)_{\alpha \in a_\xi}{}^\frown \big(\rho_*(\varepsilon, \beta)\big)_{\beta \in b_{\delta_\xi}}$$

for $\xi \in \omega_1$. Note that $\xi \in \operatorname{ran} \tau_\xi$

Fix $\nu \in \omega_1$ and to finish the proof we will find $\eta < \xi < \omega_1$ such that

$$c_0(\alpha, \beta) = \nu$$

for all $\alpha \in a_\xi$ and $\beta \in b_{\delta_\eta}$. By the choice of g, we know that there is $\eta < \xi < \omega_1$ such that

$$g(\sigma^\frown \vartheta) = \nu$$

for all $\sigma \subseteq \tau_\eta, \vartheta \subseteq \tau_\xi$ with $\eta \in \operatorname{ran}(\sigma)$ and $\xi \in \operatorname{ran}(\vartheta)$. Thus, if $\alpha \in a_\xi$ and $\beta \in b_{\delta_\eta}$ then

$$c_0(\alpha, \beta) = g(\rho_*(\alpha, \beta)) = g(\rho_*(\varepsilon, \beta)^\frown \rho_*(\alpha, \varepsilon));$$

finally observing that $\eta \in \operatorname{ran} \rho_*(\varepsilon, \beta)$ and $\xi \in \operatorname{ran} \rho_*(\varepsilon, \alpha)$ implies that

$$c_0(\alpha, \beta) = \nu.$$

$\square$

Stage IV

In this final stage, we improve the coloring of Stage III in two steps. The first modification will make it possible to realize arbitrary patterns and the second modification will yield the coloring of our Main Theorem.

Theorem 1.8. *There is a $c_1 : [\omega_2]^2 \to \omega_1$ such that for every $n \in \omega$, for every pairwise disjoint $A \in [[\omega_2]^n]^{\omega_2}$ and every $h : n \times n \to \omega_1$ there is $a < b \in A$ such that*

$$c_1(a(i), b(j)) = h(i, j)$$

for all $i, j \in n$.

Proof. Fix a 1-1 sequence $\{t_\alpha : \alpha < \omega_2\} \subseteq 2^{\omega_1}$ and let

$$\mathcal{G} = \{h : 2^d \times 2^d \to \omega_1 : d \in [\omega_1]^{<\omega}\}.$$

Fix an arbitrary bijection $\pi : \omega_1 \to \mathcal{G}$ and define $c_1 : [\omega_2]^2 \to \omega_1$ by

$$\boxed{c_1(\alpha, \beta) = h\big(t_\alpha \restriction d, t_\beta \restriction d\big)}$$

where $\boxed{h = \pi(c_0(\alpha, \beta))}$ and $\boxed{\operatorname{dom} h = 2^d \times 2^d}$.

We wish to prove that c_1 works so fix $n \in \omega$, pairwise disjoint $A \in [[\omega_2]^n]^{\omega_2}$ and $h : n \times n \to \omega_1$. Note that there is a $B \in [A]^{\omega_2}$, $d \in [\omega_1]^{<\omega}$ and $\{s_i : i < n\} \in [2^d]^n$ such that

$$t_{b(i)} \restriction d = s_i$$

for all $b \in B$ and $i < n$. Now define $\bar{h} : 2^d \times 2^d \to \omega_1$ so that $\bar{h}(s_i, s_j) = h(i, j)$ for all $i, j < n$; note that $s_i \neq s_j$ if $i \neq j$ so the definition is valid.

By the definition of c_0, there is $a < b \in B$ such that

$$\pi(c_0(a(i), b(j))) = \bar{h}$$

for all $i, j < n$. Hence

$$c_1(a(i), b(j)) = \bar{h}(t_{a(i)} \restriction d, t_{b(j)} \restriction d) = \bar{h}(s_i, s_j) = h(i, j)$$

for $i, j < n$. $\square$

Main Theorem. *There is a $c : [\omega_2]^2 \to \omega_2$ such that for every $n \in \omega$, for every pairwise disjoint $A \in [[\omega_2]^n]^{\omega_2}$ and every $h : n \times n \to \omega_2$ there is $a < b \in A$ such that*

$$c(a(i), b(j)) = h(i, j)$$

for all $i, j \in n$.

Proof. Fix a bijection $e_\alpha : \alpha \to \omega_1$ for each $\alpha < \omega_2$ and let c_1 be as in Theorem 1.8. Define $c : [\omega_2]^2 \to \omega_2$ by

$$\boxed{c(\alpha, \beta) = e_\beta^{-1}(c_1(\alpha, \beta))}$$

for all $\alpha < \beta \in \omega_2$.

We wish to prove that c works so fix $n \in \omega$, pairwise disjoint $A \in [[\omega_2]^n]^{\omega_2}$ and $h : n \times n \to \omega_2$. Let $H = \operatorname{ran} h$ and note that there is $B \in [A]^{\omega_2}$, $J_i \in [\omega_1]^{|H|}$ and $\bar{e}_i : H \to \omega_1$ for $i < n$ such that

$$\bar{e}_i = e_{b(i)} \restriction H : H \to J_i$$

for all $b \in B$ and $i < n$.

Now let $\bar{h} : n \times n \to \omega_1$ so that

$$\bar{h}(i, j) = \bar{e}_j(h(i, j))$$

for $i, j < n$. By the definition of c_1 we can find $a < b \in B$ such that

$$c_1(a(i), b(j)) = \bar{h}(i, j)$$

for $i, j < n$. Thus

$$c(a(i), b(j)) = e_{b(j)}^{-1}(\bar{h}(i, j)) = \bar{e}_j^{-1}(\bar{h}(i, j)) = h(i, j)$$

for all $i, j < n$. □

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