Sum sets and wild colourings

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The philosophy of Ramsey theory

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Goal: present a short introduction to Ramsey theory and a related additive problem.

- examples of Ramsey type theorems and certain limitations,
- turning to additive Ramsey theory,
- recent results and open problems.

Joint work with W. Weiss (U of T) and Z. Vidnyánszky (Rényi).

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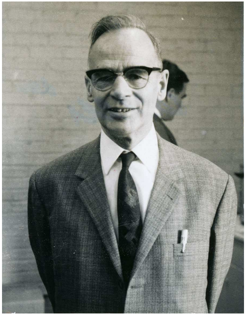

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Can we extend Ramsey's theorem to colouring pairs of $\mathbb{R}$?

[Sierpinski, 1933] There is a colouring $f : [\mathbb{R}]^2 \to \{0, 1\}$ such that $f \restriction [X]^2$ is **not constant** whenever $X \subseteq \mathbb{R}$ is **uncountable**.

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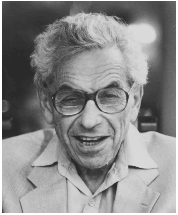

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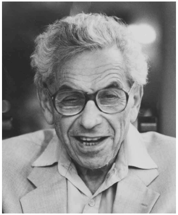

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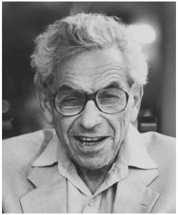

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- let $a, a + d, a + 2d, \ldots a + 2nd$ be a monochromatic arithmetic progression,
- if $x_k = \frac{a}{2} + kd$ then $x_k + x_l = a + (k + l)d$ is in the arithmetic progression for any $k, l < n$.

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Can we find an infinite X with $X+X$ is monochromatic?

Problem [J.C. Owings]: Is there a colouring $f : \mathbb{N} \to 2$ such that $X+X$ **is not monochromatic** whenever X is infinite?

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Then $X + X$ is not monochromatic whenever $X \subseteq \mathbb{N}$ is infinite.

Proof: let $X \subseteq \mathbb{N}$ be infinite, note $f(2y) = f(y) + 2$ mod 4.

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[Hindman, Leader, Strauss] Suppose that f is a finite colouring of $\mathbb{R}$. Can we find large monochromatic sum sets?

What if we suppose that the colouring is nice?

[HLS 2015] Suppose that f is a colouring of $\mathbb{R}$ with countably many colours. Suppose that

- f is measurable, or
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What if there are no analytic assumptions on the colouring?

[SWZ, Komjáth] There is a colouring $f : \mathbb{R} \to 2$ such that $f \restriction \{x + y : x \neq y \in X\}$ **is not constant** for any uncountable $X \subseteq \mathbb{R}$.

- **[HLS]** proved this using the **Continuum Hypothesis**.
- We can't realize 3 or more colours on every uncountable sum set (even more set theory comes in).

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Sums with repetitions

How about finding an infinite $X \subseteq \mathbb{R}$ with $X + X$ monochromatic?

[HLS] The CH implies that there is a colouring $f : \mathbb{R} \to 288$ such that $f \restriction X + X$ **is not constant** for **any infinite** $X \subseteq \mathbb{R}$

Recall that $\mathbb{R}$ **is a direct sum of copies of** $\mathbb{Q}$: there is $\{x_i : i \in I\} \subseteq \mathbb{R}$ so that every $x \in \mathbb{R}$ can be written uniquely as

$$x = \sum_{i \in F} c_i x_i \text{ with } c_i \in \mathbb{Q}, F \subseteq I \text{ finite.}$$

What can we say about **colouring** $\mathbb{Q}$ or finite/countable direct sums?

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From **[Hindman, Leader, Strauss]**:

- there is $f : \mathbb{Q} \to 72$ such that $f \restriction X + X$ is not constant for any infinite $X \subseteq \mathbb{Q}$;
- For any $m \in \mathbb{N}$, there is $f : \bigoplus_m \mathbb{Q} \to 72$ such that $f \restriction X + X$ is not constant for any infinite $X \subseteq \bigoplus_m \mathbb{Q}$;
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- Corollary: we can find good colourings for the first countably many cardinalities before the number of colours blows up...

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[SWV] For every finite r there is a $\kappa(r)$ so that if $f : \bigoplus_{\kappa(r)} \mathbb{N} \to r$ then there is an infinite X with $f \restriction X + X$ **constant**.

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How large is $f(n)$?

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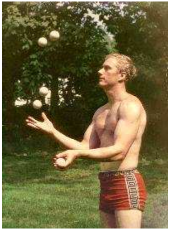

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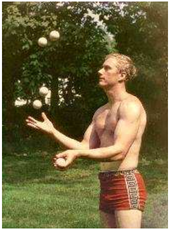

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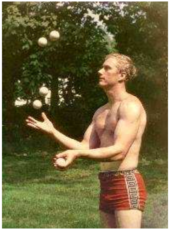

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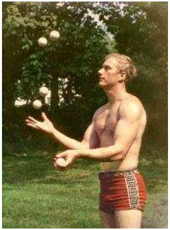

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Any questions?