Cross-like constructions and refinements

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1 Introduction

The main goal of this thesis is to investigate topologies which are originated in the idea of separate continuity. Several constructions connected to the topic are given and studied, classical topologies from the literature and new ones.

The paper splits into two parts. First general constructions on the product of two space are investigated. In the center, the most basic construction is *the cross topology*: on the product $X \times Y$ let $U \subseteq X \times Y$ be open iff every horizontal and vertical section of U is open in X and Y. Definitions and essential theorems are stated, mainly from the early and comprehensive paper of Knight, Moran and Pym [11] and Velleman [16]. Investigation of separation axioms in the cross topology shows that it will not be regular even in nice cases. In this topic the joint work of Hart and Kunen in [9] and the second part of [11], cited as [12], is fundamental. In [12] the authors define a space construction which is the completely regular analogue of the cross topology. Motivated by this paper, in Section 5 we present a new topology on $X \times Y$, denoted by $X \maltese Y$, which is much more docile, however not every time regular. The regularity of $X \maltese Y$ is characterized by a simple property for X and Y called DLP: a space X is *DLP* iff for every $x \in X$ there is a $x \in U$ neighborhood such that $cl(U) \setminus \{x\}$ is paracompact. Theorem 5.4 states:

- If (X, τ_X) and (Y, τ_Y) are T_3 and DLP, then $X \maltese Y$ is T_3.

Higher separation axioms for $X \maltese Y$ are also considered. Section 5 ends with a few words about connections between DLP and local paracompactness properties and how X being DLP is related to separation axioms in X.

The second part of the thesis generalizes first in Section 6 the idea of the radiolar topology by defining a class of spaces on $\mathbb{R}^2$. The radiolar topology, mentioned in [9] and investigated in [15] and [7], will be a special case for $S = S^1$. The S-radiolar topology for a fix $S \subseteq S^1$ is defined as follows: a set $U \subseteq \mathbb{R}^2$ is open iff for any $x \in U$ and direction $s \in S$ there is a line segment in direction s in U containing x. After listing basic properties the main result in the brief investigation is Theorem 6.8, the calculation of the character of S-radiolar topologies– by modifying a proof of Hart and Kunen. As a partial result this theorem answers the conjecture of Popvassilev [15] about the character of the radiolar topology.

- If $S \subseteq S^1$ and S contains more than one directions then the S-radiolar topology has character $2^{\mathfrak{c}}$.

We have also managed to characterize the connectedness and pathwise connectedness in the cases of S-radiolar topologies– with the aid of characterizing compact subspaces. The set $S \subseteq S^1$ is *splayed* whenever S cannot be covered in S^1 with a closed half circle. The summary of Theorem 6.15 and Corollary 6.18 is the following.

- The S-radiolar topology is connected iff S is splayed.

- The S-radiolar topology is pathwise connected iff S contains two full directions–there are $s \neq t \in S^1$ such that $\{s, -s, t, -t\} \subseteq S$

The section ends with a few words about *weak bases* and how one can show that the S-radiolar topologies and the cross topology are non homeomorphic.

Our next goal is to keep the convergence properties of the S-radiolars and regularize them. In Section 7 by merging the idea of the $X\maltese Y$ and S-radiolar topologies we define another class of spaces but these turn out to be a dead end concerning regularization. A simple modification in the definition leads us to the right class of spaces, the *the uniform $S^{\maltese}$-radiolar topologies* denoted by $\mathcal{R}(S)$. These are much more easy to handle and several topological properties are characterized throughout Section 8. The regularization is a success in some sense, by Proposition 8.4:

- The $\mathcal{R}(S)$ spaces are Tychonoff iff $S \subseteq S^1$ is closed.

We will deal with these Tychonoff cases only. After the basic observations and calculating $\chi(\mathcal{R}(S)) = \mathfrak{d}$ we introduce the evident properties for $S \subseteq S^1$: *there is no missing full direction in* S and S *contains a full direction.* Proving Lemma 8.10 shows how closely these spaces are related to the Euclidean topology in the "nice" cases.

- If in the $\mathcal{R}(S)$ topology there is no missing full direction in the defining S, then every S-open set G and its Euclidean interior can only differ in $\aleph_0$ many points.

Further investigation has done concerning Lindelöf-property in Theorem 8.14 and separability in Theorem 8.16.

- The $\mathcal{R}(S)$ spaces are hereditarily Lindelöf iff there is no missing full direction in S.
- The $\mathcal{R}(S)$ spaces are hereditarily separable iff there is no missing full direction in S.

Compact subspaces are characterized in Proposition 8.17 and in Proposition 8.18 we give necessary and sufficient conditions for the existence of uncountable compact subspaces. In Section 9 we present another modification in the definition of $\mathcal{R}(S)$ and we obtain very similar but first-countable spaces.
At the end of the thesis some open questions are stated.

2 Notations

Let S^1 denote the unit circle on the plane. We will usually mention an element $s \in S^1$ as a *direction*. A subset of $\mathbb{R}^2$ is called *Euclidean open, Euclidean closed, etc.* iff it is open, closed, etc. in the usual Euclidean topology of $\mathbb{R}^2$. For an $x \in \mathbb{R}^2, s \in S^1$ and $r > 0$ let $[x, x + rs)$ denote the half closed, half open line segment in direction s with length r which starts from x. Similar notation is used for intervals but this will not lead to misunderstanding.

Definition 2.1 *Let X, Y be arbitrary topological spaces. The* sections *of a set $U \subseteq X \times Y$ are the sets $E_a = \{y \in Y : (a, y) \in U\}$, the* vertical sections*, for any $a \in X$ and $E^b = \{x \in X : (x, b) \in U\}$, the* horizontal sections*, for any $b \in Y$.*

Let X, Y be any spaces. As we will mainly talk about different topologies simultaneously on $X \times Y$ we introduce a notation, used in [11], for the standard product: let $X\pi Y = (X \times Y, \tau_\pi)$ denote the product space on $X \times Y$.

Generally, for notions and notations which are used but undefined in the thesis see Engelking's General Topology [5].

3 The definition of the cross topology

Definition 3.1 *Let X, Y be arbitrary topological spaces. We define the topology $\tau_{X\otimes Y}$ on $X \times Y$ as follows: $U \subseteq X \times Y$ is open iff U_a is open in Y for* *all $a \in X$ and U^b is open in X for every $b \in Y$. Let $X \otimes Y = \langle X \times Y, \tau_{X\otimes Y}\rangle$ denote the above defined space, the* cross topology *on $X \times Y$.*

Notation, literature: the $X \otimes Y$ notation is from [11] and called the *tensor product* of spaces X and Y. In [16] the space $X \otimes Y$ is called the *plus topology.*

Several properties of these spaces has been investigated by many mathematicians. We gather some of the main results concerning these spaces and later on, we generalize some of them.

3.1 Basic properties of the cross topology

Evidently $X \otimes Y$ is a refinement of the space $X \pi Y$. This yields immediately that $X \otimes Y$ is Hausdorff whenever X, Y are Hausdorff. Stronger separation properties will be considered in the next section.

The basic properties of the above defined tensor product was investigated in [11] and further on in [12] by Knight, Moran and Pym . Equivalent definition by categories was given, which shows us the meaning of the notation $\otimes$, like tensor products. Let $i : X \times Y \to X \otimes Y$ denote the identity.

Definition 3.2 *The function $f : X \times Y \to Z$ is* separately continuous *iff for each $a \in X$ the function $y \mapsto f(a, y)$ is continuous and for each $b \in Y$ the function $x \mapsto f(x, b)$ is continuous.*

Proposition 3.3 ([11, Prop (1.2)]) *If $f : X \times Y \to Z$ then there is a unique function $\widehat{f} : X \otimes Y \to Z$ such that $f = \widehat{f} i$. Moreover $\widehat{f}$ is continuous iff f is separately continuous. The pair $(X \otimes Y, i)$ is determined by these properties, up to homeomorphism of $X \otimes Y$.*

As we will discuss closed discrete subspaces and density later, we cite two easy theorems:

Proposition 3.4 ([11, Proposition (4.2)]) *If $D \subseteq X \otimes Y$ and every section of D is closed and discrete then D is closed and discrete.*

Proposition 3.5 ([12, Proposition (8.1)]) *If $D \subseteq X \times Y$ such that D_e is a dense subset of Y for every $e \in E$ where E is dense in X then D is dense in $X \otimes Y$.*

In the above mentioned paper [12], the following theorem is proved:

Theorem 3.6 ([12, Proposition (8.6)]) *Let X and Y be Baire spaces and suppose that X is locally second countable. Then $X \otimes Y$ is a Baire space.*

These results are cited because they will be adequately rephrased and proved in our cases.

3.2 Axioms of separation in the cross topology

It was early known- possibly first noted by J. Novák in [14]- that $\mathbb{R} \otimes \mathbb{R}$ is not regular. This can be proved by several ways: using cardinal inequalities or by Baire-category. The paper of J. E. Hart and K. Kunen [9] seems to be the most comprehensive work about separation in the cross topology.

First, we recall a beautiful theorem from [9], which helps us to calculate the character of the spaces $X \otimes Y$ in some cases.

Theorem 3.7 ([9, Lemma 2.1]) *Suppose X, Y are T_2. Suppose that $w(X) \leq \mathfrak{c}$ and each nonempty open subset of X has size at least $\mathfrak{c}$. Suppose that there are disjoint countable dense subsets $D_\alpha \subseteq Y$ for $\alpha < \mathfrak{c}$. Then $\chi((p,q), X \otimes Y) \geq 2^{\mathfrak{c}}$ for all $(p,q) \in X \times Y$.*

This theorem immediately yields, that $\mathbb{R} \otimes \mathbb{R}$ cannot be regular; indeed for any regular space X we have $w(X) \leq 2^{d(X)}$ and the cross topology is separable by 3.5, but has character $2^{\mathfrak{c}}$. We will later apply a modified version of the proof of this theorem, to establish the character of some other spaces.

4 The complete regularization of the cross topology

The main goal of [12] is to construct the $X \widetilde{\otimes} Y$ completely regular version of $X \otimes Y$ using a topological and a category theoretical approach. From the paper we get the impression, that these constructions cannot be investigated alone well as a topological object. The positive results are mainly about cases, when $X \otimes Y = X \widetilde{\otimes} Y$.

The topological approach of the definition uses the concept of the *complete regularization of a space* X- we will not recall the definition of this, see [12] or [10]. The $X\widetilde{\otimes}Y$ is defined as the complete regularization of $X \otimes Y$. We note that if the spaces X, Y are completely regular then the underlying set of $X\widetilde{\otimes}Y$ is $X \times Y$ and we cite a category theoretical claim which is analogue to 4.1. Let $j : X\pi Y \to X\widetilde{\otimes}Y$ be the canonical map.

Proposition 4.1 ([12, Proposition (5.4)]) *If $f : X \times Y \to Z$ is separately continuous and Z is a completely regular space, then there is a unique continuous function $\widehat{f} : X\widetilde{\otimes}Y \to Z$ such that $f = \widetilde{f}j$. The pair $(X\widetilde{\otimes}Y, j)$ consisting of a completely regular space $X\widetilde{\otimes}Y$ and a separately continuous function $j : X \times Y \to X\widetilde{\otimes}Y$ is determined by this property up to homeomorphism.*

In a paper of Henriksen and Woods [10], the authors investigate the connections between four topologies on $X \times Y$ related to separate continuity-such as the cross topology, denoted by $(X \times Y, \gamma)$ and the weak topology defined by separately continuous functions, denoted by $(X \times Y, \sigma)$. Let us cite partly the following theorem, which helps to imagine the space $X\widetilde{\otimes}Y$.

Proposition 4.2 ([10, Theorem 4.8]) *Suppose X, Y are Tychonoff spaces. Then $(X \times Y, \sigma)$ is the complete regularization of $(X \times Y, \gamma)$.*

In the next section, we define a topology on $X \times Y$ which is closely related to the cross topology, has a well described neighborhood base and regularity can be guaranteed by simple assumptions on X and Y.

5 How to get regularity by the $X\maltese Y$ topology?

We will now define a topology on $X \times Y$ which refines $X\pi Y$ and coarser than $X \otimes Y$.

5.1 Defining the $X\maltese Y$ topology

Definition 5.1 *Let X and Y be arbitrary topological spaces. Define the space $X\maltese Y$ on the set $X \times Y$ with a neighborhood base for a $z \in X \times Y$ point: $\{U \subseteq X \times Y : U \setminus \{z\}$ is open in $(X \times Y) \setminus \{z\}$ with the subspace topology of $X\pi Y$ and U contains a $\tau_{X\otimes Y}$ neighborhood of $z\}$.*

From now on, if not stated otherwise open/closed/etc. means open/closed/etc. in the $\maltese$-topology.

Trivial observations about $X \maltese Y$:

- in a basic open U neighborhood of z every point has a basic product neighborhood in U, except z,
- the $X \otimes Y$ topology is always finer than $X \maltese Y$,
- the $X \maltese Y$ topology is always finer than $X \pi Y$,
- $D \subseteq X \times Y$ is dense in $X \maltese Y$ iff D is dense in $X \pi Y$.

The practicality of the definition is that several properties of $X \pi Y$ will hold also for $X \maltese Y$. As one can see, the density characters will coincide: $d(X \pi Y) = d(X \maltese Y)$. Another remark can be made about Baire-property- it is preserved by products in certain cases, for example if X is Baire and Y is (locally) second-countable and Baire.

Proposition 5.2 *If $X \pi Y$ is Baire then $X \maltese Y$ is Baire either.*

Proof: Let $G_i \subseteq X \times Y$ for $i \in \omega$ be dense open and denote $G'_i = int_\pi(G_i)$ the interior in τ_π. It can be easily seen that G'_i is dense open in $X \pi Y$, so $\cap_{i \in \omega} G'_i$ is dense in $X \pi Y$. This immediately implies that $\cap_{i \in \omega} G'_i \subseteq \cap_{i \in \omega} G_i$ is dense in $X \pi Y$ either, which is enough to be dense in $X \maltese Y$.$\square$

5.2 Axioms of separation for $X \maltese Y$

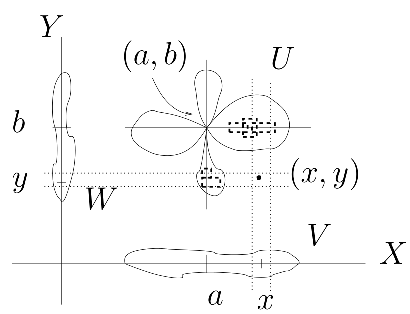
The tensor topology $X \otimes Y$ often fails to be regular and the characterization of the T_3 property is complicated. In our case the tight connection to the product topology helps us to get a simple sufficient condition for being regular.

Definition 5.3 *The space (X, τ) is said to be DLP -Dotted Locally Paracompact - iff every $x \in X$ has an open neighborhood $x \in U$ such that $cl_\tau(U) \setminus \{x\}$ is paracompact.*

Now we can formulate our main theorem about regularity in $X \maltese Y$.

Theorem 5.4 *If (X, τ_X) and (Y, τ_Y) are T_3 and DLP, then $X \maltese Y$ is T_3.*

Proof: Let $(a, b) \in X \times Y$ and $(a, b) \in U$ a basic open neighborhood– meaning that $U \setminus \{(a, b)\}$ is open in the product topology. Using the regularity in X and Y we get that there is a "cross", centered at (a, b) in U: $\{a\} \times cl_{\tau_Y}(W') \cup cl_{\tau_X}(V') \times b \subseteq U$, where $a \in V' \in \tau_X$, $b \in W' \in \tau_Y$. If any space (Z, τ) is T_3 and DLP then for every $W \in \tau$ and $x \in W$ there is a $x \in U \subseteq cl_\tau(U) \subseteq W$ such that $cl_\tau(U) \setminus \{x\}$ is paracompact; provided by the hereditary of paracompactness to closed subspaces. Thus we can suppose that $cl_{\tau_X}(V') \setminus \{a\}, cl_{\tau_Y}(W') \setminus \{b\}$ are paracompact.
There are neighborhoods $a \in V \in \tau_X$ and $b \in W \in \tau_Y$ such that $cl_{\tau_X}(V) \subseteq V'$, $cl_{\tau_Y}(W) \subseteq W'$. For every $v \in cl_{\tau_X}(V) \setminus \{a\}$ there is a $v \in G_v \in \tau_X, G_v \subseteq V' \setminus \{a\}$ and $b \in H_v \in \tau_Y$ such that the closure in the product topology $cl_\pi(G_v \times H_v) \subseteq U$. This can be done, because $(X \times Y, \tau_\pi)$ is T_3. We can further suppose that $H_v \subseteq W'$. Take $\mathcal{G} = \{G_v : v \in cl_{\tau_X}(V) \setminus \{a\}\}$, then the cover $\mathcal{G} \cup \{cl_{\tau_X}(V') \setminus cl_{\tau_X}(V)\}$ of $cl_{\tau_X}(V') \setminus \{a\}$ has a locally finite open refinement and let $\mathcal{G}'$ denote the elements which are subsets of sets in $\mathcal{G}$. For every $G \in \mathcal{G}'$ there is a $\varphi(G) \in cl_{\tau_X}(V) \setminus \{a\}$ such that $G \subseteq G_{\varphi(G)}$. Then $\{G \times H_{\varphi(G)} : G \in \mathcal{G}'\}$ covers $V \times b \setminus \{(a, b)\}$ and $\bigcup\{G \times H_{\varphi(G)} : G \in \mathcal{G}'\} \subseteq U \cap (V' \times W')$ open in τ_π.



Similarly, for every $w \in cl_{\tau_Y}(W) \setminus \{b\}$ there is a $w \in K_w \in \tau_Y, K_w \subseteq W' \setminus \{b\}$ and $a \in L_w \in \tau_X$ such that the closure in the product topology $cl_\pi(K_w \times L_w) \subseteq U$. We can suppose that $L_w \subseteq V'$. For $\mathcal{K} = \{K_w : w \in cl_{\tau_Y}(W) \setminus \{b\}\}$, the cover $\mathcal{K} \cup \{cl_{\tau_Y}(W') \setminus cl_{\tau_Y}(W)\}$ of $cl_{\tau_Y}(W') \setminus \{b\}$ has a locally finite open refinement and let $\mathcal{K}'$ denote the elements which are subsets of sets in $\mathcal{K}$. For every $K \in \mathcal{K}'$ there is a $\psi(K) \in cl_{\tau_Y}(W) \setminus \{b\}$ such that $K \subseteq K_{\psi(K)}$. Then $\{K \times L_{\psi(K)} : K \in \mathcal{K}'\}$ covers $W \times \{a\} \setminus \{(a, b)\}$ and $\bigcup\{K \times L_{\psi(K)} : K \in \mathcal{K}'\} \subseteq U \cap (V' \times W')$ open in τ_π.

Let

$$\widetilde{U} = \{(a,b)\} \cup \bigcup\{G \times H_{\varphi(G)} : G \in \mathcal{G}'\} \cup \bigcup\{K \times L_{\psi(K)} : K \in \mathcal{K}'\},$$

clearly $\widetilde{U} \subseteq U$ is an open neighborhood of (a,b).
We claim, that even $cl_\pi(\widetilde{U}) \subseteq U$. Take any $(x,y) \notin U$.
Case 1. If $x \notin cl_{\tau_X}(V')$ or $y \notin cl_{\tau_Y}(W')$ then $(x,y) \notin cl_{\tau_X}(V') \times cl_{\tau_Y}(W')$ and $cl_\pi(\widetilde{U}) \subseteq cl_{\tau_X}(V') \times cl_{\tau_Y}(W')$ gives us $(x,y) \notin cl_\pi(\widetilde{U})$.
Case 2. Suppose that $x \in cl_{\tau_X}(V')$ and $y \in cl_{\tau_Y}(W')$. Than $x \neq a$, $y \neq b$ and there is an $x \in M \in \tau_X$ neighborhood such that M only intersects finitely many elements of $\mathcal{G}'$, these are $\mathcal{G}'' = \{G_1, ..., G_m\} \subseteq \mathcal{G}'$. We can suppose that $a \notin M$. Similarly there is a $y \in N \in \tau_Y$ such that N only intersects $\mathcal{K}'' = \{K_1, ..., K_n\}$ from $\mathcal{K}'$. We will use the following notations:

$$A = cl_\pi(\{(a,b)\} \cup \bigcup\{G \times H_{\varphi(G)} : G \in \mathcal{G}' \setminus \mathcal{G}''\})$$

$$B = cl_\pi(\bigcup\{K \times L_{\psi(K)} : K \in \mathcal{K}' \setminus \mathcal{K}''\})$$

$$C = \bigcup_{G \in \mathcal{G}''} cl_\pi(G \times H_{\varphi(G)}) \cup \bigcup_{K \in \mathcal{K}''} cl_\pi(K \times L_{\psi(K)}).$$

Clearly $cl_\pi(\widetilde{U}) = A \cup B \cup C$. Evidently $C \subseteq U$, so $(x,y) \notin C$. On the other hand, $A \subseteq (X \setminus M) \times Y$ product closed, so $(x,y) \notin A$. Similarly $B \subseteq X \times (Y \setminus N)$ product closed, so $(x,y) \notin B$. Thus $(x,y) \notin cl_\pi(\widetilde{U})$ which gives us: $cl(\widetilde{U}) \subseteq cl_\pi(\widetilde{U}) \subseteq U$. $\square$

Being DLP is not a necessary condition, because if X is discrete then $X \maltese Y = X \pi Y$. The next claim shows that this local property is needed, meaning that even in the nice case of X being compact, $X \maltese X$ can fail to be regular.

Proposition 5.5 *$(\omega_1 + 1) \maltese (\omega_1 + 1)$ is not T_3*

Proof: We claim that $x_0 = (\omega_1, \omega_1)$ cannot be separated from the closed set $\Delta = \{(\alpha, \alpha) : \alpha < \omega_1\}$. Indeed $\Delta \cup \{x_0\}$ is closed in the product topology and $x_0 \in (\omega_1 + 1) \times (\omega_1 + 1) \setminus \Delta$ is an open neighborhood of x_0 disjoint from Δ, thus Δ is closed. Take any open neighborhood U of Δ. For every limit ordinal $\alpha < \omega_1$ there is a $f(\alpha) < \alpha$ such that $(f(\alpha), \alpha] \times \{\alpha\} \subseteq U$. By the Fodor

Pressing Down Lemma, there is a $\alpha < \omega_1$ and ordinals $\{\alpha_\xi : \xi < \omega_1\}$ such that $f(\alpha_\xi) = \alpha$ for all $\xi < \omega_1$. This means that $\bigcup_{\xi<\omega_1}((f(\alpha), \alpha_\xi] \times \{\alpha_\xi\}) \subseteq U$, thus the set $(f(\alpha), \omega_1] \times \{\omega_1\}$ is in the closure of U, so x_0 cannot be separated from Δ. $\Box$

Regularity was deduced from simple separation-like axioms. However, T_4 property will not hold in $X\maltese Y$ without restrictions to cardinalities.

Lemma 5.6 *Let X, Y arbitrary spaces, $E \subseteq X \times Y$ closed in τ_π and E_x, E^y discrete in the corresponding topology, for every $x \in X, y \in Y$. Then E is closed discrete in $X\maltese Y$.*

Proof: One readily checks that $\{(x, y)\} \cup ((X \times Y) \setminus E)$ is open and only contains (x, y) from E. From E_x, E^y being discrete, the previous neighborhood trivially contains a $\tau_{X\otimes Y}$ neighborhood of (x, y).$\Box$

For example, for any Hausdorff X the set $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$ is closed discrete in $X\maltese X$ with cardinality $|X|$. Thus we get the following easy corollary.

Corollary 5.7 *If the Hausdorff space X is separable and has cardinality $\mathfrak{c}$ then $X\maltese X$ cannot be normal.*

Now it is natural to ask, when will the space $X\maltese Y$ be Tychonoff? The assumption made in the next proposition is rather strong and made for the product, thus it is most desirable to be weakened.

Proposition 5.8 *If $X\pi Y$ is hereditarily normal, then $X\maltese Y$ is Tychonoff.*

Proof: Let $(a, b) \in U \subseteq X\maltese Y$, an open neighborhood of (a, b). There is a $a \in V_a \in \tau_X$ such that $cl_{\tau_X}(V_a) \times \{b\} \subseteq U$ and $b \in V_b \in \tau_Y$ such that $\{a\} \times cl_{\tau_Y}(V_b) \subseteq U$. Let $F = cl_{\tau_X}(V_a) \times \{b\} \cup \{a\} \times cl_{\tau_Y}(V_b)$. Then in $X\pi Y \setminus \{(a, b)\}$ the set $F \setminus \{(a, b)\}$ is closed and $int_\pi(U) \setminus \{(a, b)\}$ is a product open neighborhood of it, so there is a $F\setminus\{(a, b)\} \subseteq U' \subseteq cl_\pi(U') \subseteq U\setminus\{(a, b)\}$ π-open neighborhood. So there is an f continuous function $X\pi Y \setminus \{(a, b)\} \to [0, 1]$ such that $f|_{cl_\pi(U')} \equiv 0$ and $f|_{X\pi Y\setminus U} \equiv 1$. Let $f((a, b)) = 0$. This makes f $\maltese$-locally zero at (a, b), thus continuous on the whole $X\maltese Y$. $\Box$

Remark: By Morita [13] it is sufficient to assume that X is metrizable and Y is perfectly normal, these assumptions imply the hereditary normality of $X\pi Y$.

5.3 Shortly on DLP spaces

The assumption made in 5.4 on X and Y being T_3 is needed, meaning that even DLP and T_2 does not imply regularity.

Example: Let $X = [0, 1)$. The τ topology on X is defined by neighborhood bases: every point in $(0, 1)$ has Euclidean neighborhoods and 0 has the next neighborhood base: $\{[0, a) \setminus D : a \in (0, 1)$, D is a Euclidean closed discrete set in $(0, 1)\}$. These neighborhoods generate a topology. We claim that this space is evidently T_2 and a bit less evidently not regular but DLP. Property DLP holds for every point in $(0, 1)$ and a dotted neighborhood of 0 has the Euclidean topology as the subspace topology, so X is DLP. If $F \subseteq X$ is a Euclidean closed discrete set, which converges to 0 in the Euclidean sense, then it is a closed set in τ but the closure of any neighborhood of 0 intersects F, so X is not regular.

However, DLP spaces are "almost" Tychonoff:

Proposition 5.9 *If (X, τ) is a DLP space then there is a closed discrete $Z \subseteq X$ such that $X \setminus Z$ is Tychonoff.*

Proof: For any $x \in X$, let $x \in U_x \in \tau_X$ such that $cl_{\tau_X}(U_x) \setminus \{x\}$ is paracompact. Then for $U = \cup\{U_x \setminus \{x\} : x \in X\}$, the set $Z = X \setminus U$ is closed and discrete, provided by the U_x neighborhoods. Let $y \in U_x \setminus \{x\}$ and V be an arbitrary neighborhood of y. There are $W_1, W_2 \in \tau_X$ such that $y \in W_1 \subseteq cl_{\tau_X}(W_1) \subseteq W_2 \subseteq cl_{\tau_X}(W_2) \subseteq V \cap U_x \setminus \{x\}$. $cl_{\tau_X}(W_2)$ is paracompact, so Tychonoff, so there is a $f : cl_{\tau_X}(W_2) \to [0, 1]$ such that $f(y) = 0$ and f is 1 at $cl_{\tau_X}(W_2) \setminus W_1$. Define f to be 1 on $X \setminus cl_{\tau_X}(W_2)$. The extended f is continuous and separates y and $X \setminus V$. $\square$

The space $\omega_1 + 1$ with the order topology shows, that not even compactness implies DLP. How DLP is related to other basic local paracompactness properties?

Definition 5.10

- *The space (X, τ) is* LP- Locally Paracompact- *iff for every $x \in X$ there is a $X \in U \in \tau$ such that $cl_\tau(U)$ is paracompact.*
- *The space (X, τ) is* LHP- Locally Hereditarily Paracompact- *iff for every $x \in X$ there is a $x \in U \in \tau$ such that $cl_\tau(U)$ is hereditarily paracompact.*

Proposition 5.11 *If the space $\langle X, \tau\rangle$ is LHP then it is DLP. Conversely, if X is DLP and T_3 then X is LP either.*

Proof: The first state is trivial. For the second claim, suppose that for an $x \in X$ and a neighborhood $x \in U \in \tau$ the set $cl_\tau(U) \setminus \{x\}$ is paracompact. Then $cl_\tau(U)$ is paracompact either. Indeed, suppose $\mathcal{G}$ is an open cover of $cl_\tau(U)$. Let $\widetilde{G} \in \mathcal{G}$ with $x \in \widetilde{G}$ then there is a $x \in V \subseteq cl_\tau(V) \subseteq \widetilde{G}$ neighborhood either. Then $\mathcal{G}' = \{G \setminus cl_\tau(V) : G \in \mathcal{G}\} \cup \{\widetilde{G} \setminus \{x\}\}$ is an open cover of $cl_\tau(U) \setminus \{x\}$ so $\mathcal{G}'$ has a locally finite open refinement $\mathcal{G}''$. Then $\mathcal{G}'' \cup \{\widetilde{G}\}$ is a locally finite open cover of $cl_\tau(U)$ which refines $\mathcal{G}$.$\square$

6 The S-radiolar topologies

From now on, we will consider spaces which has the underlying set $\mathbb{R}^2$. Using the next construction of Brown [4], we can define some further topologies:

Definition 6.1 *Let (Z, τ) be a topological space and $\mathcal{E}$ any family of subsets of X. Then $\tau_\mathcal{E}$ is the family of all $U \subseteq Z$ such that $U \cap E$ is open in E in the subspace topology, for every $E \in \mathcal{E}$.*

For example, if $(Z, \tau) = X\pi Y$ and $\mathcal{E} = \{\{x\} \times Y, X \times \{y\} : x \in X, y \in Y\}$ then the $\tau_\mathcal{E}$ topology is the above defined $\tau_{X\otimes Y}$ tensor. Another example, which is mentioned in [9], if we take $\mathbb{R}^2$ with the Euclidean topology and $\mathcal{E}$ to be all the lines in $\mathbb{R}^2$. This topology was called the *radiolar topology* and investigated by Popvassilev [15], Roman Fric in [7]. Fric proved that every topology between the cross and radiolar topology has sequential order ω_1. For the cross topology, this fact was showed by G. H. Greco in [8].

The goal of this section is to introduce and investigate a bit more general family of topological spaces.

Definition 6.2 *Let $S \subseteq S^1$. An S-*star at *x is a set $\varphi^x \subseteq \mathbb{R}^2$ where $x \in \mathbb{R}^2$, $\varphi : S \to \mathbb{R}^+$ and $\varphi^x = \bigcup\{[x, x + \varphi(s)s) : s \in S\}$.*

Let $\mathbf{St}(x, S)$ denote all the S-stars at a point x.

Definition 6.3 *For every $S \subseteq S^1$ we will define the S-*radiolar topology *on $\mathbb{R}^2$ as follows: $U \subseteq \mathbb{R}^2$ is said to be* S-radiolar open *iff for every point $x \in U$ there is an S-star at x in U. The S-radiolar topology is the set of all S-radiolar open sets and noted by τ_S.*

6.1 Basic properties of the S-radiolar topologies

The analogy of the cross topology and the S-radiolar topology is trivial, the cross topology on $\mathbb{R}^2$ is a special case of the S-radiolar topologies.

Convergence: Take a sequence in $\mathbb{R}^2$: (x_n), which converges to a point x in the S-radiolar topology. It can easily be checked that this means exactly, that (x_n) converges to x from finitely many directions which are in S. Precisely, there is an $N \in \omega$ such that the directions $\{\overrightarrow{xx_n}\}_{n \geq N}$ is a finite subset of S. In some sense this topology generalizes the Sorgenfrey-line, where $x_n \to x$ iff x_n converges to x from the right side (or left side, depends on the definition).

Trivial observations about special cases:

- when $S = S^1$ we get back the above mentioned radiolar-topology,
- when S is the adequate four point set we get $\mathbb{R} \otimes \mathbb{R}$ as the S-radiolar topology,
- when $|S| = 1$ the S-radiolar topology is $\mathfrak{c}$ many disjoint Sorgenfrey-lines.
- when $S = \{s, -s\}$ the S-radiolar topology is $\mathfrak{c}$ many disjoint Euclidean lines.

Proposition 6.4 *For $S \subseteq T \subseteq S^1$, τ_S refines τ_T. If $S \neq T$ then we obtain a proper refinement.*

Proof: The first claim is trivial, every T-radiolar open is S-radiolar open. If $t \in T \setminus S$, then the set we get if we leave out a half line in direction t from the plane, is S-radiolar open but not T-radiolar open. $\square$

Further on we will assume that **for all defining** $S \subseteq S^1$ **sets** $S \nsubseteq \{s, -s\}$ **for any** $s \in S^1$ and we need to make an other assumption for the sake of simplicity. Namely, we can suppose that every time there are some directions in S which are parallel to the standard axis' of the plane. This helps us to benefit further on from the E_a and E^b notions, because if E is S-radiolar open then these sets will have nonempty Euclidean interior.

Separation properties: As we refined the Euclidean topology, the S-radiolars are Hausdorff. However, they will not be regular. For proving this, we will prove an analogue to the following statement of Velleman:

Theorem 6.5 ([16, Theorem 6]) *Suppose that U is nonempty and cross open, $U = \cup_{n\in\omega} U_n$. Then for some n, the cross closure of U_n has nonempty Euclidean interior.*

We modify this and get the following stronger property for the S-radiolars than being Baire:

Proposition 6.6 *Suppose that U is nonempty and S-radiolar open, $U = \cup_{n\in\omega} U_n$. Then for some n, the S-radiolar closure of U_n has nonempty Euclidean interior.*

Proof: Take a point $(a, b) \in U$ and an $\varepsilon > 0$ such that $[a, a+\varepsilon) \subseteq U^b$ and $[b, b+\varepsilon) \subseteq U_a$. This can be done according to our assumption on S from the beginning of the section. For $x \in [a, a+\varepsilon)$ the set $U_x = \cup_{n\in\omega}(U_n)_x$ must contain an interval, so there is an n_x for which $cl((U_{n_x})_x)$ contains an interval by the Baire Category Theorem. Thus there are p_x, q_x rational numbers such that the interval $(p_x, q_x) \subseteq cl((U_{n_x})_x)$. For each $n \in \omega$ and p, q rational numbers $X_{n,p,q} = \{x \in [a, a+\varepsilon) : n = n_x, p = p_x, q = q_x\}$. By the Baire Category Theorem again, there must be some n, p, q such that $cl(X_{n,p,q})$ has nonempty interior, there are $c < d$ such that the interval $(c, d) \subseteq cl(X_{n,p,q})$. It follows immediately that $(c, d) \times (p, q) \subseteq cl(U_n)$. $\square$

Thus we are able to prove the statements about regularity:

Proposition 6.7 *Every S-radiolar topology is not regular.*

Proof: Take a set F on the plane which is dense in the Euclidean sense and has only two points on each line. Such a set can be constructed easily by transfinite induction. Then F is S-radiolar closed, take any $x \notin F$. If U is an S-radiolar neighborhood of x then it has a nonempty S-radiolar closure in $\mathbb{R}^2$ so it contains a point of F. This shows that the S-radiolar topology cannot be regular. $\square$

6.2 The character of the S-radiolar topologies

Here we present the modified proof of Lemma 2.1 from [9], which was stated above as 3.7 and which helps us to calculate the exact character of the S-radiolar topologies. S. G. Popvassilev in [15] conjectured that the cross and radiolar topology has character and weight $2^{\mathfrak{c}}$. This more general result to S-radiolar topologies answers this question.
As it will help to follow the proof, we will use the notation $\mathbb{R}_1$ for the real line which is the horizontal axis and $\mathbb{R}_2$ for the vertical axis.

Theorem 6.8 *For $S \subseteq S^1$ the S-radiolar topology has character $2^{\mathfrak{c}}$.*

Proof: If not stated otherwise, we mean open/closed/etc. sets S-radiolar open/closed/etc.
It is sufficient to find $F_\delta \subseteq \mathbb{R}^2$, $\delta < 2^{\mathfrak{c}}$ closed discrete subsets such that every countable union $\cup_{n\in\omega} F_{\delta_n}$ is dense, for distinct δ_n. In fact, if a (p, q) would have character less than $2^{\mathfrak{c}}$ then we could find a W neighborhood of (p, q) and distinct δ_n such that $W \cap (F_{\delta_n} \setminus \{(p, q)\}) = \emptyset$. Since $\cup_{n\in\omega} F_{\delta_n}$ is dense, (p, q) would be isolated, which is a contradiction.
We construct disjoint sets $\{B_\alpha\colon \alpha < \mathfrak{c}\}$ in $\mathbb{R}_1$ such that each $|B_\alpha| = \mathfrak{c}$ and for all nonempty Euclidean open set U there is an α such that $B_\alpha \subseteq U$. Indeed, take an enumeration of all open sets $\{U_\alpha : \alpha < \mathfrak{c}\}$. Take the pairs $\langle U_\alpha, \beta\rangle$ for $\alpha, \beta < \mathfrak{c}$ and by transfinite induction choose an $x^\alpha_\beta \in U_\alpha$ distinct from the ones we have choosen before. The sets $B_\alpha = \{x^\alpha_\beta : \beta < \mathfrak{c}\}$ will satisfy our conditions.
The following lemma will be needed, which can be found in the paper of Engelking and Karlowicz [6].

Lemma 6.9 *If the sets $\{B_\alpha \subseteq \mathbb{R} : \alpha < \mathfrak{c}\}$ are disjoint and have size $\mathfrak{c}$ then there are $g_\delta : \mathbb{R} \to \omega$, $\delta < 2^{\mathfrak{c}}$ such that for each $\alpha < \mathfrak{c}$ the sequence $\langle g_\delta | B_\alpha : \delta < 2^{\mathfrak{c}}\rangle$ is σ-independent: for $n \in \omega$ given distinct $\delta_n < 2^{\mathfrak{c}}$ and $k_n \in \omega$ there is a $x \in B_\alpha$ such that $g_{\delta_n}(x) = k_n$.*

Next we construct D_x disjoint, countable, Euclidean dense subsets of $\mathbb{R}_2$ for each $x \in B = \cup_{\alpha<\mathfrak{c}} B_\alpha$. By using a similar transfinite induction to the one in the Mazurkiewich theorem, one can assure that the set $D = \{(x, y) : x \in B, y \in D_x\}$ has at most two points on each non vertical line.
Index each D_x as $\{d^n_x\}_{n\in\omega}$. Define $F_\delta : B \to \mathbb{R}_2$ by $F_\delta(x) = d_x^{g_\delta(x)}$. We will use the notation F_δ for the graph of this function as well. Since the D_x sets are

disjoint, F_δ is 1-1 and by the construction of D the graph of F_δ is S-radiolar closed and discrete.
Now fix distinct δ_n for $n \in \omega$, and take the set $H = \cup_{n\in\omega} F_{\delta_n}$. To show that H is dense, fix a nonempty $N \subseteq \mathbb{R}^2$ open set and we show that $N \cap cl(H) \neq \emptyset$. Fix $y \in \mathbb{R}_2$ such that $N^y \neq \emptyset$. Then N^y has a nonempty Euclidean interior, which contains a B_α. There is a $x \in B_\alpha$ such that $g_{\delta_n}(x) = n$ for all $n \in \omega$. Thus $F_{\delta_n}(x) = d^n_x$, so $\{x\} \times D \subseteq H$, so $\{x\} \times \mathbb{R}_2 \subseteq cl(H)$, so $(x, y) \in N \cap cl(H)$.$\square$

6.3 Connectedness in the S-radiolar topologies

In the cross topology, the question of connectedness is nearly trivial:

Theorem 6.10 ([11, Theorem (4.1)]) *The components and path-wise components of $X \otimes Y$ are the same sets as those of $X\pi Y$.*

However, in our case with S-radiolars, we might have problems. The sections of a set as subspaces- Sorgenfrey-line, discrete space- are totally disconnected several times. As we will see, the characterization of connected S-radiolar topologies can be nicely done and we will obtain a similar result in these cases to the previously cited one.

Definition 6.11 *Take three line segments starting from one point-the center - on the plane, with the same length. Fix one segment, which we will call the main one. If the angles between the main and the other segments are greater than $\pi/2$ but less than π than we call this set a λ-scheme.*

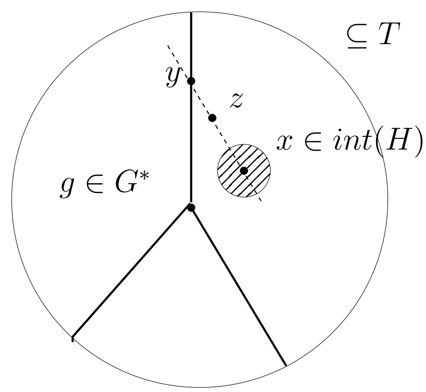
As a Λ_0 λ-scheme is a special S_0-star determined by a three point set $S_0 \subseteq S^1$, we will use the notation $(\mathbb{R}^2, \tau_{\Lambda_0})$ for the corresponding S_0-radiolar topology. For any $x \in \mathbb{R}^2$ and $r > 0$ let $\Lambda_0(x, r)$ denote the Λ_0 λ-scheme centered at x with radius r.

Definition 6.12 *An $S \subseteq S^1$ set is "splayed" iff no closed half-circle contains it.*

For example every λ-scheme is splayed, meaning that the defining three point subset from S^1 is splayed. It can easily be seen, that if S is not splayed then the S-radiolar topology is not connected- one can make a partition of $\mathbb{R}^2$ into a Euclidean open and a closed half plane, which are both open in the S-radiolar topology.

Theorem 6.13 *For an arbitrary Λ_0 λ-scheme, in the $(\mathbb{R}^2, \tau_{\Lambda_0})$ topology any Euclidean connected, Euclidean open set is Λ_0-radiolar connected.*

Proof: Let $T = G \cup H$ be an Euclidean open, Euclidean connected set, where G, H are nonempty radiolar open sets, we want to get a contradiction. Define $G^* = G \setminus int(G)$ and $H^* = H \setminus int(H)$ as described, the points which have no Euclidean neighborhood in G and H.
(1) We prove, that $G^* \subseteq (H^*)'$ and $H^* \subseteq (G^*)'$: the two cases are symmetric, we will deal with $G^* \subseteq (H^*)'$. From the definition we get $G^* \subseteq (H)'$ but suppose that only $int(H)$ accumulates to a point: $g \in G^*$. So there is a $r > 0$ such that $B(g, r) \cap H^* = \emptyset$. One can suppose that for this r: $B(g, r) \subseteq T$ and $\Lambda_0(g, r) \subseteq G$. It can easily be seen that there is a $x \in int(H) \cap B(g, r)$, so close to g that there is a line passing x in a direction which is in Λ_0, intersecting $\Lambda_0(g, r) \subseteq G$ in a $y \in G$ point. This can be made in a way, that the subspace appearing on the line is a Sorgenfrey-line, with the open set $[y, x)$. For this, we used the property of being splayed.

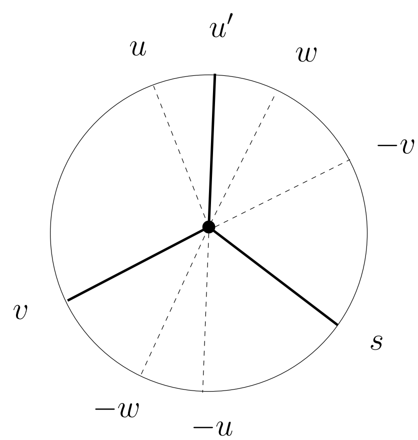


Consider the supreme of points in G towards x on the $[y, x)$ half line, let this be: $z \in T$. Now z cannot be in H, because than it would be in $int(H)$, but z is also an accumulation point of G. If z is in G, than there is a λ-scheme in G with center z, and that would intersect the $[y, x)$ half line in a half open interval which would indicate- because of the criteria towards the subspace topology- that there would be more points in G towards x. This contradicts to the fact, that z was a supreme point. Thus, we proved: $G^* \subseteq (H^*)'$.
(2) $G^*, H^* \neq \emptyset$ and $G^* \cup H^*$ is Euclidean closed: they cannot be both empty, because T is Euclidean connected, but from (1) then the other one is nonempty either. Being Euclidean closed is trivial.

(3) The contradiction. Choose an $x_1 \in G^*$, which exists because of (2). $\Lambda_0(x_1; r_1) \subseteq G$ and for this r_1 we can suppose that $B(x_1, r_1) \subseteq T$. There exists a $r_1 > s_1 > 0$ such that for any $x \in B(x_1; s_1) \cap H^*$ the maximal λ-scheme in H with center x has radius at most $r_1/2$, or else this λ-scheme would intersect the λ-scheme in G with center x_1- we have again used the property of being splayed. Take an $x_2 \in B(x_1; s_1) \cap H^*$, which exists because of (1) and take a neighborhood $\Lambda_0(x_2; r_2) \subseteq H \cap B(x_1; s_1)$ where we get $r_2 < r_1/2$, because of our assumptions on s_1. There is an $r_2 > s_2 > 0$ such that for every $x \in B(x_2; s_2) \cap G^*$ every λ-scheme in G with center x has radius at most $r_2/2$ or else this would intersect the λ-scheme in H with center x_2. The following is straightforward: take $x_3 \in B(x_2; s_2) \cap G^*$ and $\Lambda_0(x_3; r_3) \subseteq G \cap B(x_2; s_2)$, $r_3 < r_2/2 < r_1/4$. We continue with induction and this way we construct a Cauchy-sequence. So there is a limit point: $x \in G^* \cup H^*$, because $G^* \cup H^*$ is Euclidean closed. However, x cannot be in G^* nor in H^*. This is because, there is no λ-scheme in H nor in G with center x with positive radius, because $x \in B(x_i, s_i)$ which forbids that this radius could be greater than $r_1/2^i$.$\square$

Lemma 6.14 *If S is splayed and contains no full direction then S contains a λ-scheme (a three point subset that determines a λ-scheme as a star).*

Proof: Take a $v \in S$ and the half circle bounded by the vector orthogonal to v. It has to contain a $u \in S$, without loss of generality we can suppose that the $\widehat{uv}$ arc is shorter then $\widehat{vu}$. Take the supreme of the directions in S from u towards $-v$- in negative direction, let this be w. $w \neq -v$ because S contains no full direction.



S intersects the open half circle which is bounded by w and in negative direction from w, so there is a $s \in S$ as the figure shows. One can easily see, that if we choose a $u' \in S$ close to w-as it is a supreme, this will exist- we will have a λ-scheme with u', v, s, where the position of s decides that which segment is the main one.$\Box$

Theorem 6.15 *The S-radiolar topology is connected iff S is splayed. If S is splayed then every Euclidean open, Euclidean connected set is S-radiolar connected.*

Proof: If S is not splayed then as the previous remark shows the S-radiolar topology is not connected.
If S is splayed and contains a full direction then every $G \subseteq \mathbb{R}^2$ Euclidean open, Euclidean connected set is S-radiolar connected. Let $G = R \cup T$ be any partition to S-radiolar open sets where $R \neq \emptyset$, let $s \in S$ be any full direction. Then no line in direction s can be cut into two parts by R and T, so if $x \in R$ then the whole line passing x in direction s is in R. Combining this with the fact that R is S-radiolar open, where S is splayed one can see that there is a Euclidean neighborhood of $x \in R$ in R. Similarly arguing about T, it is Euclidean open either. Thus one of them is empty- provided by G being connected.
If S is splayed and contains no full direction then by the previous lemma there is a λ-scheme in S, so the S-radiolar topology eventually can be refined to a connected topology, thus by Theorem 6.13 every Euclidean open, Euclidean connected set is S-radiolar connected.$\Box$

Remark: One can imagine easily an S, which contains no full directions however splayed– λ-schemes for example. For that, the S-radiolar topology is connected but all the lines as subspaces are totally disconnected.

Our next goal is to deal with pathwise connectedness. The following lemma is the analogue of Theorem (4.3) from [11] for S-radiolar topologies.

Lemma 6.16 *If $K \subseteq \mathbb{R}^2$ is S-radiolar compact then K can be covered by finitely many lines.*

Proof: Suppose on the contrary that there are no finite family of lines that contain the compact K. By induction one can easily construct an infinite

$D \subseteq K$ such that every line intersects D in at most two points. Such a set is S-radiolar closed and discrete which contradicts the fact that K is compact. $\square$

Corollary 6.17 *The S-radiolar topology contains an uncountable compact subset iff S contains a full direction– there is an $s \in S^1$ such that $s, -s \in S$.*

Proof: If S contains a full direction the Euclidean line appears as a subspace. The other part follows easily from the previous lemma and from the fact that in the Sorgenfrey-line or in the discrete topology the compact subsets are countable– and these topologies appear on lines as subspaces if there is no full direction in S.$\square$

Corollary 6.18 *The S-radiolar topology is pathwise connected iff there are two full directions in S– there are $s \neq t \in S^1$ such that $\{s, -s, t, -t\} \subseteq S$.*

Proof: If there are two full directions $s, t \in S^1$ in S then the S-radiolar topology is pathwise connected; the Euclidean continuous function $f : I \to \mathbb{R}^2$ is S-radiolar continuous if $f(I)$ is a subset of the union of a line in direction s and in direction t. Any two points can be connected with such functions.
Suppose that the S-radiolar topology is pathwise connected, this yields that there is at least one full direction $s \in S$ because $f(I)$ is compact with cardinality $\mathfrak{c}$. Let $f : I \to \mathbb{R}^2$ be S-radiolar continuous connecting two different points– which are not on a line in direction s. We claim that if s is the only full direction in S then $f(I)$ can be covered by finitely many lines $f(I) \subseteq \cup_{i=1\ldots n} L_i$ and L_i is in direction s, which is a contradiction– the inverse images of the sets $f(I) \cap L_i$ would be a partition of I to open-and-closed sets. If $f(I)$ cannot be covered in such way then in any cover from Lemma 6.16 there is a line L which is not in direction s and contains infinitely many points of $f(I)$. The subspace topology on L indicates that $|f(I) \cap L| = \omega$. Thus there would be an isolated point $x \in f(I)$ and $\emptyset \neq f^{-1}\{x\} \neq I$ would be an open-and-closed set in I which is a contradiction.$\square$

6.4 Weak bases

As mentioned above, the cross topology is a special case of the S-radiolar topologies. Also, from the previous claims about connectedness one can easily deduce that there are sets $S \subseteq S^1$ such that the S-radiolar topology and the cross topology are not homeomorphic. The next definition is due to A. V. Arhangelskii– can be found in [1]– and will be needed in the further investigation.

Definition 6.19 *For a space (X, τ) and a point $x \in X$ a family of sets $\mathcal{B}$ from X is a* weak base at x *iff*

- $x \in \bigcap \mathcal{B}$,
- *for every $x \in U \subseteq X$ the set U is open iff $U \setminus \{x\}$ is open and there exists a $B \in \mathcal{B}$ such that $B \subseteq U$.*

Now it is straightforward to generalize first-countability. While [1] deals with this local concept, we also need the global definition of a weak base– the first appearance of the following definition is probably in [2].

Definition 6.20 *The space (X, τ) is* weakly first-countable *iff for every $x \in X$ there is a countable family of closed sets $\mathcal{B}_x$ such that:*

- $x \in \bigcap \mathcal{B}_x$,
- *$U \subseteq X$ is open iff for every $x \in U$ there is a $B \in \mathcal{B}_x$ such that $B \subseteq U$.*

Clearly for any first-countable space X, Y the space $X \otimes Y$ is weakly first-countable.
In the introduction of [7] the author claims that the cross topology on $\mathbb{R}^2$ is weakly first-countable while the radiolar topology is not- thus they cannot be homeomorphic. The following lemma is easy to prove and leads us to this result.

Lemma 6.21 *If $\{B_i\}_{i \in I}$ is a weak base at $x \in \mathbb{R}^2$ for the S-radiolar topology then there is weak base at x consisting of stars $\{C_i\}_{i \in I} \subseteq \mathbf{St}(x, S)$ such that $C_i \subseteq B_i$.*

Informally the S-stars at x form "the smallest" weak base at x in the S-radiolar topology. Using this lemma and an easy diagonal argument we have the following:

Proposition 6.22 *For any infinite $S \subseteq S^1$, the S-radiolar topology and the cross topology on $\mathbb{R}^2$ are not homeomorphic.*

Remark: There are several definitions in the literature for weak bases and weak first-countability, even the same authors use different definitions in different papers.

As we have seen, every weak base at a point for an S-radiolar topology can be refined to a weak base consisting of S-stars. It would be interesting to see exactly how many S-stars are needed? Namely, consider the following cardinal function.

Definition 6.23 *Let X be any space, $x \in X$. The* weak base character of x in X *is $\chi_w(x, X) = min\{|\mathcal{B}| : \mathcal{B}$ is a weak base at $x\}$.*

We will not use the name *weak character* because it was used by Arhangelskii and Buzyakova for another definition in [3]. Let $\chi_w(x, (\mathbb{R}^2, \tau_S)) = \chi_w(S)$ as the S-radiolar topologies are homogeneous. We have seen that $\chi_w(S)$ is greater than ω if S is infinite– conversely if S is finite, the weak base character is ω trivially and $2^{\mathfrak{c}}$ is an upper bound of course for all cases. Easy to see that another upper bound is if we take dominating families. Let us recall some definitions, notions.

- For any cardinal κ let ${}^{\kappa}\omega$ denote the set of all functions $f : \kappa \to \omega$.
- A family of functions $\mathcal{F} \subseteq {}^{\kappa}\omega$ is *dominating in* ${}^{\kappa}\omega$ iff for every $g \in {}^{\kappa}\omega$ there is a $f \in \mathcal{F}$ such that $g(\alpha) < f(\alpha)$ for all $\alpha < \kappa$.
- $\mathfrak{d}(\kappa) = min\{|\mathcal{F}| : \mathcal{F}$ is dominating in ${}^{\kappa}\omega\}$.

It is straightforward that $\mathfrak{d}(n) = \omega$ for all $n \in \omega$ and $\mathfrak{d}(\omega) = \mathfrak{d}$ is the well known *dominating number*. For all κ we have $\kappa < \mathfrak{d}(\kappa) \leq 2^{\kappa}$. By using the previously cited result Lemma 6.9 of Engelking and Karlowicz one can show, that $\mathfrak{d}(\mathfrak{c}) = 2^{\mathfrak{c}}$. These cardinalities will not change if we take $\mathbb{R}$ or $\mathbb{R}^+$ to be the range of the dominated functions.
Using these notions we can formulate the following equality.

Proposition 6.24 *For any $S \subseteq S^1$ we have $\chi_w(S) = \mathfrak{d}(|S|)$.*

Proof: Suppose that the set $\{\varphi_i : S \to \mathbb{R}^+\}_{i\in I}$ is dominating for all functions $\varphi : S \to \mathbb{R}$. Then the set of S-stars $\{(\frac{1}{\varphi_i})^x\}_{i\in I}$ is a weak base at x because every set of S-stars can be refined by this set. Thus $\chi_w(S) \leq \mathfrak{d}(|S|)$. Suppose we have a weak base of stars at x, let this denote $\{(\frac{1}{\varphi_i})^x\}_{i\in I}$ where $\varphi_i : S \to \mathbb{R}^+$. We will show that this is a dominating family, thus $\mathfrak{d}(|S|) \leq \chi_w(S)$. Take any function $\varphi : S \to \mathbb{R}^+$ and enumerate $S = \{s_\alpha : \alpha < \kappa = |S|\}$. By transfinite induction construct the set $F = \{x_\alpha : \alpha < \kappa\}$ such that $x_\alpha \in (x, x + \frac{1}{2\varphi(s_\alpha)} s_\alpha)$ and every line intersects F in at least two points. F is closed discrete and $x \in U = \mathbb{R}^2 \setminus F$ thus there is an $i \in I$ such that $(\frac{1}{\varphi_i})^x \subseteq U$. So we have $\frac{1}{\varphi_i(s_\alpha)} < \frac{1}{2\varphi(s_\alpha)}$ for all $\alpha < \kappa$ meaning that $\varphi < \varphi_i$ everywhere on S.□

As a stronger version of Proposition 6.22 we get the following corollary.

Corollary 6.25 *If $S_i \subseteq S^1$ for $i = 1, 2, 3$, where $|S_1| < \omega$, $|S| = \omega$ and $|S| = \mathfrak{c}$ then the S_i-radiolar topologies are non homeomorphic.*

Proof: The weak base characters are

$$\chi_w(S_1) = \omega < \chi_w(S_2) = \mathfrak{d} < \chi_w(S_3) = 2^{\mathfrak{c}}.$$

Thus they cannot be homeomorphic.□

6.5 Symmetrizability

We were able to make differences between S-radiolar topologies when the defining S sets have entirely different cardinalities. In this section we will show that even for quite "similar" S sets the S-radiolar topologies can be non homeomorphic. This will be done by using the following weakening of metrizability.

Definition 6.26 *The space X is* symmetrizable *iff there is a $d : X \times X \to \mathbb{R}$* symmetric on X, *meaning that*

1. *for all $x, y \in X$: $d(x, y) = d(y, x) \geq 0$,*
2. *$d(x, y) = 0$ iff $x = y$,*

3. *the set $U \subseteq X$ is open iff for every $x \in U$ there is a $\varepsilon > 0$ such that $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\} \subseteq U$.*

Not surprising that this property will appear in cases when the defining set S of the S-radiolar topology is *symmetric*: $s \in S$ iff $-s \in S$.

Proposition 6.27 *For an $S \subseteq S^1$ the S-radiolar topology is symmetrizable iff the set S is finite and symmetric.*

Proof: Suppose S is symmetric. For any $x, y \in \mathbb{R}^2$ define $d(x, y) = |x - y|$ the Euclidean distance if x and y are on a line in a direction $s \in S$, otherwise let $d(x, y) = 1$. This is clearly a symmetric and the sets $B(x, \varepsilon)$ for $\varepsilon > 0$ are stars at x forming a weak base at x for finite S. Thus the S-radiolar topology is symmetrizable.
Suppose the S-radiolar topology is symmetrizable. Every symmetrizable space is weakly first-countable thus $|S| < \omega$. It suffices to prove the following:

1. Let (X, τ) be symmetrizable with a d symmetric, $C \subseteq X$ a closed subspace. Then $(C, \tau|_C)$ is a symmetrizable space with the symmetric $d|_{C \times C}$.

2. The Sorgenfrey-line is not symmetrizable.

If 1. and 2. holds then S must be symmetric or else there is a Sorgenfrey-line as a closed subspace.
Proof of 1.: straightforward.
Proof of 2.: suppose that d is a symmetric on the Sorgenfrey-line L; the topology is generated by the $[a, b)$ intervals. Then there is a rational $\varepsilon_x > 0$ for all $x \in L$ such that $B(x, \varepsilon) \subseteq [x, \infty)$. There is a set $L' \subseteq L$ with $|L'| = \aleph_1$ such that $\varepsilon_x = \varepsilon$ for all $x \in L'$. Then for all $x, y \in L'$ we have $x \notin B(y, \varepsilon)$. There is a decreasing sequence $\{x_n\}_{n \in \omega} \subseteq L'$ such that $x_n \to x \in L'$. Then the set $A = L \setminus \{x_n\}_{n \in \omega}$ is not open however $B(x, \varepsilon) \subseteq A$ and the other points have Euclidean neighborhoods in A thus it should be open by the definition of a symmetric.$\square$

This proposition allows us to make difference between S-radiolar topologies which only have a slight difference in the defining S sets. The next proposition shows that the number of full directions in finite S defining sets (thus in the symmetrizable cases the cardinality of the defining S sets) is a further invariant.

Proposition 6.28 *Let $S, T \subseteq S^1$ be finite and $\sigma = |\{s \in S : -s \in S\}|$ and $\tau = |\{t \in T : -t \in T\}|$ such that $\sigma \neq \tau$. Then the S-radiolar and T-radiolar topology are not homeomorphic.*

Proof: Suppose $\sigma < \tau$. It suffices to prove the following:

1. for all $x \in \mathbb{R}^2$ the S-stars at x form a weak base for the S-radiolar topology such that for all $y \in B \in \mathbf{St}(x, S)$ the set $B \setminus \{y\}$ has at most 2σ connected components which has cardinality $\mathfrak{c}$,

2. for all $x \in \mathbb{R}^2$ and $\mathcal{C}_x$ weak base at x for the T-radiolar topology there is a $C \in \mathcal{C}_x$ such that $C \setminus \{x\}$ has 2τ connected components with cardinality $\mathfrak{c}$.

We omit the proof as it is straightforward.□

When S is finite, the symmetrizability of the S-radiolar topology characterized the existence of Sorgenfrey-lines as closed subspaces. For arbitrary $S \subseteq S^1$ sets to see the difference between symmetric and not symmetric cases one shall investigate "minimal" weak bases.

Proposition 6.29 *Let $S \subseteq S^1$ be symmetric, $T \subseteq S^1$ not symmetric. Then the S-radiolar and T-radiolar topology are not homeomorphic.*

Proof: It suffices to show the following:

1. for every $x \in \mathbb{R}^2$ and weak base at x $\mathcal{B}_x$ for the S-radiolar topology, there is a refinement $\mathcal{B}'_x$ of $\mathcal{B}_x$ (take S-stars by Lemma 6.21) such that every element of $\mathcal{B}'_x$ is connected,

2. for every $x \in \mathbb{R}^2$ there is a weak base at x $\mathcal{C}_x$ for the T-radiolar topology (take T-stars by Lemma 6.21) such that for every refinement $\mathcal{C}'_x$ of $\mathcal{C}_x$ there is a $C \in \mathcal{C}'_x$ which is not connected.

Proofs can be carried out easily, so we will not present it here. □

7 The $S^{\maltese}$-radiolars

The next definition adjoins the idea of the S-radiolars with the above investigated $\maltese$-topology in favor of getting regular spaces.

Definition 7.1 *The $S^{\maltese}$-radiolar topology: fix a $S \subseteq S^1$. A $U \subseteq \mathbb{R}^2$ is said to be $S^{\maltese}$-radiolar open iff for every point $x \in U$ there is a V S-radiolar neighborhood of x in U such that $V \setminus \{x\}$ is Euclidean open.*

Such V sets will be called *basic $S^{\maltese}$-radiolar neighborhoods.*

In some cases the $S^{\maltese}$-radiolar topology must be regular, moreover Tychonoff. For example, when we get back the original $\maltese$-topology from the specific four point S– proposition 5.8 gives us this result. Our investigation shows two things. First a positive result.

Proposition 7.2 *If $S \subseteq S^1$ is countable then the $S^{\maltese}$-radiolar topology is regular.*

Proof: Let $x \in U$ be a basic open neighborhood. Let $S = \{s_n\}_{n\in\omega}$. For every direction $s_n \in S$ there is $r_n > 0$ and a Euclidean open U_n neighborhood of the segment $(x, x + r_n s_n)$ in U. Let $r'_n = min\{\frac{1}{n}, \frac{r_n}{2}\}$. There is a V_n Euclidean open such that $(x, x + r'_n s_n) \subseteq V_n \subseteq cl(V_n) \subseteq U_n \cap B(x, \frac{2}{n})$. Then $V = \{x\} \cup \bigcup_{n\in\omega} V_n$ is $S^{\maltese}$-radiolar open and it is easy to check that $cl(V) \subseteq U$.$\square$

This second proposition shows, why we will not go into any further details about these topologies, but investigate some nicer ones- which will be defined in the upcoming section. We will locally use the following notation: if $a, b \in S^1$ let $\widehat{a, b}$ denote the closed arc between a and b, the open arc $int(\widehat{a, b})$.

Proposition 7.3 *If $S \subseteq S^1$ contains a nonempty perfect set $P \subseteq S$ then the $S^{\maltese}$-radiolar topology is not regular.*

Proof: Let $x \in \mathbb{R}^2$ be any point. Suppose that the distinct points $\{s_n\}_{n\in\omega} \subseteq P$ are dense in P. Let $U = \mathbb{R}^2 \setminus \{x_n = x + \frac{1}{n}s_n : n \in \omega\}$ then $x \in U$ and U is $S^{\maltese}$-radiolar open. Let $x_n \in G_n$ be any open neighborhood. We claim that if $x \in H$ and for $G = \cup_{n\in\omega} G_n$ we have $G \cap H = \emptyset$ then H cannot be open.

Let $n_0 \geq 2$. There are $a_0, b_0 \in S^1$ such that $s_{n_0} \in int(\widehat{a_0, b_0})$ and for all $s \in \widehat{a_0, b_0}$ the line segment $[x, x + \frac{1}{n_0 - 1}s) \cap G_{n_0} \neq \emptyset$. Straightforward to construct by induction $n_0 < n_1 < ... < n_k < ...$ and $a_k \neq b_k$ such that for $j < k : s_{n_j} \neq s_{n_k} \in int(\widehat{a_k, b_k})$ and $[x, x + \frac{1}{n_k - 1}s) \cap G_{n_k} \neq \emptyset$ for all $s \in \widehat{a_k, b_k}$ and the arcs form a decreasing system. We can further suppose that $\cap_{k \in \omega} \widehat{a_k, b_k} = \{p\}$. By P being perfect and $s_{n_k} \in P \cap \widehat{a_k, b_k}$ we have $p \in P \subseteq S$. However by the definition of $\widehat{a_k, b_k}$ arcs there is no line segment from x in direction p which would not intersect the appropriate G_{n_k} set.$\square$

The claim is about a fairly huge class of S sets. Thus not even for $S = S^1$, the $S^{\text{✠}}$-radiolar topology is not going to be regular.

8 The uniform S-radiolars and $S^{\text{✠}}$-radiolars

These last modifications in the definitions lead us to one of the main interest of this paper. This way, we obtain such topologies which are in some sense- and some cases- near to the Euclidean topology however they have similar convergence properties as the S-radiolars.

Definition 8.1 *Fix a $S \subseteq S^1$, $x \in \mathbb{R}^2, r > 0$. Let us consider the following special S-stars:*

$$S(x, r) = \bigcup \{[x, x + rs) : s \in S\}.$$

Definition 8.2 *Let $S \subseteq S^1$. The* uniform S-radiolar topology *is defined as follows: an $U \subseteq \mathbb{R}^2$ is said to be* uniform S-radiolar open *iff for every point $x \in U$ there is a $r > 0$ such that $S(x, r) \subseteq U$. The uniform S-radiolar topology is the collection of all uniform S-radiolar open sets.*

Definition 8.3 *Let $S \subseteq S^1$. The* uniform $S^{\text{✠}}$-radiolar topology *is defined as follows: an $U \subseteq \mathbb{R}^2$ is said to be* uniform $S^{\text{✠}}$-radiolar open *iff for every point $x \in U$ there is a $x \in V \subseteq U$ such that $S(x, r) \subseteq V$ for some $r > 0$ and $V \setminus \{x\}$ is Euclidean open. The collection of all uniform $S^{\text{✠}}$-radiolar open sets form the uniform $S^{\text{✠}}$-radiolar topology.*

For example, for finite S sets the uniform S-radiolar is simply the S-radiolar topology. For open S sets, the uniform S-radiolars and uniform $S^{\text{✠}}$-radiolars coincide. As we will mainly talk about uniform $S^{\text{✠}}$-radiolars we will use the short notation $\mathcal{R}$ for this space. From now on S-open, S-closed sets are the open, closed sets in the $\mathcal{R}(S)$ topology.

8.1 Basic properties of $\mathcal{R}(S)$

Convergence: Take a sequence in $\mathbb{R}^2$: (x_n), which converges to a point x in the $\mathcal{R}(S)$ topology. It can easily be checked that this means that the sequence is almost covered by any $S(x, r)$.

Separation axioms: The $\mathcal{R}(S)$ spaces are Hausdorff but not every time regular- however characterization of regularity is simple.

Proposition 8.4 *The $\mathcal{R}(S)$ spaces are Tychonoff iff $S \subseteq S^1$ is closed. If S is not closed then $\mathcal{R}(S)$ is not even regular.*

Proof: Suppose $s_i \in S$ and $s_i \to s \notin S$. For $S' = S^1 \setminus \{s\}$, the set $S'(x, r)$ is S-open but for every $x \in U$ S-open neighborhood, the S-closure of U has a line segment in it in direction s.
If S is closed we will use the hereditary normality of $\mathbb{R}^2$ with the Euclidean topology. Let $x \in U$ be any S-open neighborhood of x. There is an $r > 0$ such that $F = cl(S(x, r)) \setminus \{x\} \subseteq U \setminus \{x\}$. F is Euclidean closed in $\mathbb{R}^2 \setminus \{x\}$. There is a V Euclidean open such that $F \subseteq V \subseteq cl(V) \subseteq U$ and a Euclidean continuous function $f : \mathbb{R}^2 \setminus \{x\} \to [0, 1]$ such that $f|_{cl(V)} \equiv 0$ and $f|_{\mathbb{R}^2 \setminus U} \equiv 1$. If $f(x) = 0$, then the new function is S-locally zero at x, so continuous.$\square$

In the following theorems, we will concentrate on these nice cases, when the $\mathcal{R}(S)$ topologies are Tychonoff.

Lines as subspaces: We make some easy observations on lines as subspaces of $\mathcal{R}(S)$: for a fixed $s \in S^1$ direction there can be one of the following three kinds of topologies on a line in direction s:

- the Euclidean topology iff $s, -s \in S$,
- the Sorgenfrey-line iff only one of s and $-s$ is in S,
- the discrete topology iff $s, -s \notin S$.

Corollary 8.5 *The Sorgenfrey-line or the $\mathfrak{c}$-discrete space is a subspace besides the trivial cases of $S = \emptyset$ or $S = S^1$ so the $\mathcal{R}(S)$ spaces are not metrizable.*

Corollary 8.6 *If $S \neq \emptyset$ then $\mathcal{R}(S)$ is separable, so cannot be normal when there is a missing full direction in S because then there is a closed discrete space with cardinality $\mathfrak{c}$.*

Remark: These observations on lines as subspaces also hold for S-radiolar topologies of course.

Connectedness: As the S-radiolar topology refines the $\mathcal{R}(S)$ space for every S, the question of connectedness is clear by Theorem 6.15.

Corollary 8.7 *Let $S \subseteq S^1$ be closed. The $\mathcal{R}(S)$ space is connected iff S is splayed. If S is splayed then every Euclidean open, Euclidean connected set is connected in $\mathcal{R}(S)$ either.*

8.2 The character of $\mathcal{R}(S)$

The character of the $\mathcal{R}(S)$ spaces is more closely related to a certain dominating number than the character or weak base character of the S-radiolar topologies.

Proposition 8.8 *For every nonempty closed $S \subsetneq S^1$: $\chi(\mathcal{R}(S)) = \mathfrak{d}$.*

Proof: The space is homogeneous so we just have to determine the character in an arbitrary point. From the definition of the neighborhoods, it is sufficient to determine the character of the set $S(x,r) \setminus \{x\}$ in $\mathbb{R}^2 \setminus \{x\}$ in the Euclidean topology. Applying a homeomorphism- moving x to "infinity"- one can see that our task is to determine the character of an F closed subset of the Euclidean plane, where $bd(F)$ is unbounded. We claim that this is $\mathfrak{d}$. Let $\mathcal{D}$ be a dominating family in ω^ω, consisting of non zero functions. Take K_n compact subsets of F such that: $F = \cup_{n\in\omega} K_n$ and every $x \in F$ is just in finitely many K_n. We define a family of open sets: $\mathcal{B} = \{B_f : f \in \mathcal{D}\}$ such that $B_f = \cup\{B(x, \frac{1}{f(n)}) : x \in F, n = min\{m \in \omega : x \in K_m\}\}$. We prove that this is a neighborhood base for F, for this let $F \subseteq U$ be an arbitrary open set. Then for every K_n there is a $g(n) \in \omega$ such that $B(K_n, \frac{1}{g(n)}) \subseteq U$. Than there is a $f \in \mathcal{D}$ such that $g(n) \leq f(n)$. In this case $F \subseteq B_f \subseteq U$: if $y \in B_f$ than there is a $x \in F$ such that $y \in B(x, \frac{1}{f(n)})$ where $x \in K_n$. So $y \in B(x, \frac{1}{f(n)}) \subseteq B(x, \frac{1}{g(n)}) \subseteq B(K_n, \frac{1}{g(n)}) \subseteq U$.
Suppose that $\mathcal{B} = \{B_\alpha : \alpha < \kappa\}$ is a neighborhood base for F. We will

show that $\kappa \geq \mathfrak{d}$ and for this, we will construct a dominating family. Let $\{x_n\}_{n\in\omega}$ be a closed discrete subset of $bd(F)$ with disjoint neighborhoods $x_n \in U_n$. For each $n \in \omega$ there is a sequence $\{y_k^n\}_{k\in\omega} \subseteq U_n \setminus F$ converging to x_n. We define a family of functions: $\mathcal{D} = \{f_\alpha \in \omega^\omega : \alpha < \kappa\}$ such that for each $\alpha < \kappa$ and $n \in \omega$: $f_\alpha(n) = max\{m \in \omega : \forall k \leq m : y_k^n \notin B_\alpha\}$. Let $g \in \omega^\omega$ be an arbitrary function. Consider the $x_n \in V_n \subseteq U_n$ open neighborhoods for each $n \in \omega$, such that $y_k^n \notin V_n$ for every $k \leq g(n)$. The open set $U = \mathbb{R}^2 \setminus \{x_n, y_k^n : n, k \in \omega\} \cup \bigcup_{n\in\omega} V_n$ is a neighborhood of F, so there is a $\alpha < \kappa$: $B_\alpha \subseteq U$. For this α we have $g(n) \leq f_\alpha(n)$ for every $n \in \omega$. $\square$

8.3 $\mathcal{R}(S)$ and the Euclidean topology

It is time to define two basic properties which will help us to make the connections between S and topological properties of $\mathcal{R}(S)$.

Definition 8.9 *For an $S \subseteq S^1$ we say, that* there is no missing full direction in S, *iff $x \notin S \Rightarrow -x \in S$ and* there is a full direction in S *iff there is a $x \in S^1$ such that $-x \in S^1$ either.*

We will say, that $x \in U$ is a *S-neighborhood of x with radius r* if U is S-open and $S(x, r) \subseteq U$.

Lemma 8.10 *If in the $\mathcal{R}(S)$ topology there is no missing full direction in the defining S, than every S-open set G and its Euclidean interior can only differ in $\aleph_0$ many points.*

Proof: For every S-open G let $G^* = G \setminus int(G)$. We prove that G^* is countable, that is only countable many points do not have Euclidean neighborhoods in G. Every point in G has an S-neighborhood in G, fix one for every point with rational radius. These only intersect G^* in their center, every other point of an S-neighborhood is part of the Euclidean interior. If $|G^*| > \aleph_0$ than there are more than countably many points in G^* with S-neighborhoods in G with a fix $r > 0$ rational radius. These points have an Euclidean accumulation point, we only use the fact that there are $x, y \in G^*$ such that $|x - y| < r/10$. There are no missing full directions in S, so the xy or the yx direction is in S. This implies that $y \in B_S(x, \varepsilon, r) \subseteq G$ or $x \in B_S(y, \varepsilon, r) \subseteq G$, which is a contradiction.$\square$

We get the following easy statement:

Proposition 8.11 *In $\mathcal{R}(S)$ the number of open sets is $\mathfrak{c}$ iff there is no missing direction in the defining set S. If there is a missing direction there are $2^{\mathfrak{c}}$ open sets.*

Proof: If there is a missing direction then in such a direction the lines as subspaces have the discrete topology. If there is no missing direction in S, then we can use the 8.10 Lemma.$\square$

Proposition 8.12 *If there is no missing full direction in $S \subseteq S^1$ Euclidean closed then*

- *if $s \in S$ and $-s \notin S$ then $s \in int(S)$,*
- *there is a full direction in S.*

Proof: Suppose that $s \in S \setminus int(S)$. Then there is a sequence $s_i \in S^1 \setminus S$ such that $s_i \to s$. Then $-s_i \in S$ and $-s_i \to -s$ so provided by S being closed, $-s \in S$.
By the previous claim, if there is no full direction in the closed S then $S = int(S)$ would be a clopen set. $\square$

Corollary 8.13 *If there is no discrete subspace on any line in $\mathcal{R}(S)$ then there are lines which have the Euclidean topology on them.*

8.4 Lindelöf property, separability

Theorem 8.14 *The $\mathcal{R}(S)$ spaces are hereditarily Lindelöf iff there is no missing full direction in S.*

Proof: If there is a missing full direction, than we have a discrete subspace with cardinality $\mathfrak{c}$.
For the other implication let $X \subset \mathbb{R}^2$ be an arbitrary set, $X \subseteq \bigcup\{G_i : i \in I\}$ S-open cover. By using 8.10 Lemma for $\cup_{i \in I} G_i$, we obtain that there are only countably many points in X, which are not covered by any G_i's interior: let these points be H. For the $X \setminus H \subseteq \bigcup\{int(G_i) : i \in I\}$, Euclidean open cover we can use the Lindelöf-theorem. So we have a countable subcover:

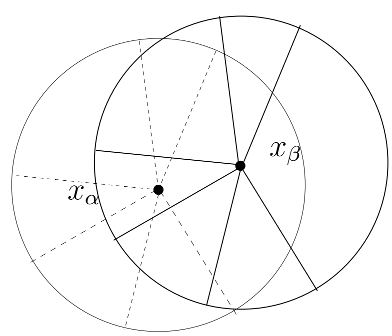
$X \setminus H \subseteq \bigcup\{G_j\}_{j\in\omega}$. For every $h \in H$ there is a G_h from the original cover such that $h \in G_h$. H is countable, so $\{G_j\}_{j\in\omega} \cup \{G_h\}_{h\in H}$ is a countable subcover for X. □

Corollary 8.15 *The $\mathcal{R}(S)$ spaces are normal iff there is no missing full direction in S, indeed T_3+Lindelöf $\Rightarrow T_4$.*

While the $\mathcal{R}(S)$ spaces are trivially separable, a stronger claim can be made.

Theorem 8.16 *The $\mathcal{R}(S)$ spaces are hereditarily separable iff there is no missing full direction in S.*

Proof: If there is a missing full direction then we have a discrete subspace with cardinality $\mathfrak{c}$.
Let $X \subseteq \mathbb{R}^2$ be an arbitrary subspace. Take one point $x_1 \in X$. We can suppose that this is not a dense subspace. With transfinite recursion we define H: if H is still not dense in X, take a new point from an open set in X, which is disjoint from H-,for limit ordinals just take unions. Now, for every $x_\alpha \in H$ there is a G_α S-neighborhood- with rational radius- such that $\beta < \alpha$-ra $x_\beta \notin G_\alpha$. If we choosed ω_1 many points this way then there are more than countable many points with S-neighborhoods with a fixed r radius. These points form a set $\widehat{H}$. This set has two condensation points: $x_\alpha, x_\beta \in \widehat{H}$, closer than $r/10$, where we can suppose that: $\alpha > \beta$.



In this case $x_\beta \notin G_\alpha$, but $x_\alpha \in G_\beta$, because they are close to each other and there is no missing full direction. In this case, the direction $s = (x_\beta x_\alpha) \in S$ is eventually in the interior of S, because of Proposition

8.12. x_β is a condensation point either, so there is an $x_\gamma \in \widehat{H}$, $\gamma > \alpha$, such that $x_\alpha \in G_\gamma$ because G_γ is just a translate of G_β, closely to direction $s \in int(S)$. This contradicts to the choice of G_γ.$\Box$

8.5 Compactness, compact subspaces

While trivially the whole space will not be compact– nor sequentially, countably compact– from the previous section it is straightforward that $\mathcal{R}(S)$ will be paracompact iff there is no missing full direction in S.

First, we would like to characterize the compact subspaces with some easy conditions. As it can be seen on the Sorgenfrey-line, being bounded and (S-)closed is not enough for compactness.

Proposition 8.17 *A $K \subseteq \mathbb{R}^2$ subspace is S-compact iff it is Euclidean compact and for every $x \in K$ there is a $r > 0$ such that $K \cap B(x,r) \subseteq S(x,r)$.*

Proof: Let K be an S-compact subspace. Trivially K is Euclidean compact either. Suppose that there is an $x \in K$ such that for every $n \in \omega$ there is an $x_n \in K$ such that $x_n \in B(x, \frac{1}{n}) \setminus S(x, \frac{1}{n})$. Then $\{x, x_n : n \in \omega\}$ is S-closed and a subspace of K, so it should be S-compact. Though, it has the discrete topology.
Now suppose that K satisfy the conditions from above and $K = \bigcup_{i\in\Gamma} G_i$ is an arbitrary S-open cover, we can suppose that the open sets are S-neighborhoods. Let G^* be the points, which are only covered by centers. If $|G^*|$ is infinite, than it has an Euclidean accumulation point: $k \in K$. There is an $r > 0$, that the points closer to k than r are in an S-neighborhood of k which is part of the G_i covering k. So these points cannot be in G^*, so G^* is finite. These points can be covered by finitely many G_i's. After this, we just need to choose a finite subcover for this covering: $K \setminus G^* = \bigcup_{i\in\Gamma} int(G_i)$. This can be done, because this is a Euclidean open cover of a Euclidean compact set.$\Box$

Interesting fact, that on the Sorgenfrey-line there are no uncountable compact subspaces. There are certain $\mathcal{R}(S)$ spaces where one can trivially find uncountable compact spaces- if S contains a full direction, Euclidean lines appear.

Proposition 8.18 *There exists an uncountable $K \subseteq \mathbb{R}^2$ S-compact subspace iff there is a full direction in S*

Proof: If there is a full direction in S then there are Euclidean lines as subspaces.
Now suppose, that K is an S-compact subspace such that $|K| > \aleph_0$. Because of 8.17 Proposition, for every $x \in K$ there is an $r^{(x)} \in \mathbb{Q}$ such that $K \cap B(x, r^{(x)}) \subset S(x, r^{(x)})$. There are more than countably many points with the same rational r, their set is $H \subset K$. There are two accumulation points in H: x, y such that they are closer than $r/10$. Thus directions xy and yx are both in S. $\square$

This proposition gives us a partial result concerning pathwise connectedness in $\mathcal{R}(S)$ spaces: if the $\mathcal{R}(S)$ is path-wise connected then the defining $S \subseteq S^1$ must contain a full direction.

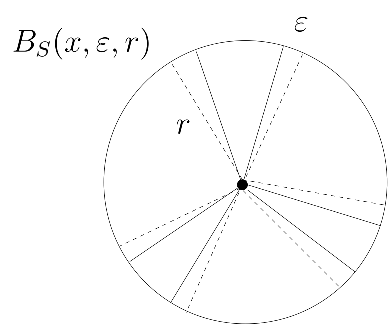
9 A modification for first-countability

We will give a definition of a topology on $\mathbb{R}^2$ which nearly coincides with $\mathcal{R}(S)$, but has the advantage of being first-countable. Let $B(S, \varepsilon)$ note the ε radius Euclidean neighborhood of the set S in S^1.

Definition 9.1 *Let $S \subseteq S^1$. For every $x \in \mathbb{R}^2$ and $\varepsilon, r > 0$:*

$$B_S(x, \varepsilon, r) = \bigcup \{[x, x + rs) : s \in B(S, \varepsilon)\}$$

is the S-neighborhood of x with radius r and width ε.



Definition 9.2 *For any $S \subseteq S^1$ Euclidean-closed let $\widehat{\mathcal{R}}(S) = (\mathbb{R}^2, \widehat{\tau}_S)$ be the following topology: $G \subseteq \mathbb{R}^2$ is open iff for every $x \in G$ there are $\varepsilon, r > 0$ such that $B_S(x, \varepsilon, r) \subseteq G$.*

By this modification we trivially get a first-countable space. The price is that the convergence in $\widehat{\mathcal{R}}(S)$- as convergence from directions in S- will be a little less "sharp": $(x_n) \to x$ iff for every $\varepsilon, r > 0$ the sequence is almost in $B_S(x, \varepsilon, r)$- this allows a convergent sequence to avoid $S(x, r)$ completely. Besides these two differences the propositions, theorems and remarks made for $\mathcal{R}(S)$ will hold for $\widehat{\mathcal{R}}(S)$ either- only minor changes are needed.

10 Open problems

In this last section, we would like to pose some open questions concerning these topics.

10.1 Axioms of separation for $X \maltese Y$

We have gave sufficient conditions for X and Y which made the space $X \maltese Y$ regular. The hereditary normality of the product space $X \pi Y$ makes $X \maltese Y$ Tychonoff. Our main interest is that do we need such strong properties of the product to get stronger separation axioms for $X \maltese Y$?

Question 10.1 *Are there any "reasonable" conditions for X and Y that makes the space $X \maltese Y$ Tychonoff?*

10.2 Differentiating the topologies

We have defined several classes of spaces and investigated in detail two main classes, the S-radiolar and $\mathcal{R}(S)$ topologies. We have seen that for finite and infinite S sets, the S-radiolars cannot be homeomorphic. It is natural to ask:

Question 10.2 *How many non homeomorphic S-radiolar topologies have we defined?*

Another, maybe more ambitious task is to answer the following question.

Question 10.3 *For $S, S' \subseteq S^1$ when will the S-radiolar and S'-radiolar topologies coincide?*

Lindelöf property, compact subspaces or connectedness in $\mathcal{R}(S)$ spaces shows that there are non homeomorphic spaces among these, but such properties separate just finitely many cases. However, in this case we have the following partial result.

Proposition 10.4 *Let $S, T \subseteq S^1$ be closed sets. Suppose S and T are splayed and have different finite number of connected components. Then $\mathcal{R}(S)$ and $\mathcal{R}(T)$ are not homeomorphic.*

Proof: Suppose S has s and T has t many connected components, $s < t$. The assumptions yield that S and T are just finite union of closed intervals. It suffices to show the following:

1. there is a base $\mathcal{B}$ for $\mathcal{R}(S)$ such that there is a unique point x in every $U \in \mathcal{B}$ such that $U \setminus \{x\}$ is the union of s many Euclidean open, connected sets thus by 8.7 it has exactly s many S-connected components– and leaving other points will not increase the number of connected components,

2. in every base $\mathcal{C}$ for $\mathcal{R}(T)$ there is set $C \in \mathcal{C}$ and a point $y \in C$ such that $C \setminus \{y\}$ has at least t connected components.

As there is no theoretical difficulty in proving these claims– just technical problems, describing precisely the neighborhoods– we omit the proof.□

Thus, there are at least countably infinitely many different $\mathcal{R}(S)$ spaces– choosing S_i to be a set with i points ($i \in \omega$), the $\mathcal{R}(S_i)$ spaces are non homeomorphic. However there are $\mathfrak{c}$ closed subsets in S^1, so the analogue problems to the previous ones are still open.

Question 10.5 *How many non homeomorphic $\mathcal{R}(S)$ topologies have we defined?*

Question 10.6 *For closed $S, S' \subseteq S^1$ when will the $\mathcal{R}(S)$ and $\mathcal{R}(S')$ spaces coincide?*

10.3 Pathwise connectedness in $\mathcal{R}(S)$

Similarly to S-radiolar topologies if S contains two full directions then $\mathcal{R}(S)$ will be pathwise connected. This was a necessary condition in the case of S-radiolars. However, that proof cannot be applied here, because compact subspaces of $\mathcal{R}(S)$ are cannot be covered always by finitely many lines– just consider the case $S = S^1$.

Question 10.7 *Is it necessary to S contain at least two full directions to the space $\mathcal{R}(S)$ be pathwise connected?*

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