Coloring problems on infinite graphs

Dániel T. Soukup

University of Toronto

Miami University, January 23 2015

The outline of our problems

- we work with infinite graphs: **countably or uncountably** many vertices;
- edge-coloring problems: Ramsey-type results and **partitions into monochromatic subgraphs**;
- vertex-coloring problems: **structural properties** of graphs with **large chromatic number**,
- some **problems** I would like to solve.

The outline of our problems

- we work with infinite graphs: **countably or uncountably** many vertices;
- edge-coloring problems: Ramsey-type results and **partitions into monochromatic subgraphs**;
- vertex-coloring problems: **structural properties** of graphs with **large chromatic number**,
- some **problems** I would like to solve.

The outline of our problems

- we work with infinite graphs: **countably or uncountably** many vertices;
- edge-coloring problems: Ramsey-type results and **partitions into monochromatic subgraphs**;
- vertex-coloring problems: **structural properties** of graphs with **large chromatic number**,
- some **problems** I would like to solve.

The outline of our problems

- we work with infinite graphs: **countably or uncountably** many vertices;
- edge-coloring problems: Ramsey-type results and **partitions into monochromatic subgraphs**;
- vertex-coloring problems: **structural properties** of graphs with **large chromatic number**,
- some **problems** I would like to solve.

The outline of our problems

- we work with infinite graphs: **countably or uncountably** many vertices;
- edge-coloring problems: Ramsey-type results and **partitions into monochromatic subgraphs**;
- vertex-coloring problems: **structural properties** of graphs with **large chromatic number**,
- some **problems** I would like to solve.

Motivation

- **anti-Ramsey theory**:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- **chromatic number problems**:
  - theory of **expanders**,
  - applications in **computer science**;

Motivation

- **anti-Ramsey theory**:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- **chromatic number problems**:
  - theory of **expanders**,
  - applications in **computer science**;

Motivation

- **anti-Ramsey theory**:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- **chromatic number problems**:
  - theory of **expanders**,
  - applications in **computer science**;

Motivation

- **anti-Ramsey theory**:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- **chromatic number problems**:
  - theory of **expanders**,
  - applications in **computer science**;

Motivation

- **anti-Ramsey theory**:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- **chromatic number problems**:
  - theory of **expanders**,
  - applications in **computer science**;

Motivation

- **anti-Ramsey theory**:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- **chromatic number problems**:
  - theory of **expanders**,
  - applications in **computer science**;

Motivation

- anti-Ramsey theory:
  - applications in **general topology**: L-spaces,
  - applications in **functional analysis**: Banach-spaces and free sequences;
- chromatic number problems:
  - theory of **expanders**,
  - applications in **computer science**;

Theorem (R. Rado, 1978)

If the edges of the complete graph on $\mathbb{N}$ are colored with finitely many colors then the vertices can be ***covered by disjoint monochromatic paths of different color****.*

P. Erdős on Richard Rado:

"I was good at discovering perhaps difficult and interesting special cases, and Richard was good at generalizing them and putting them in their proper perspective."

Edge-colored complete graphs

The origins

Theorem (R. Rado, 1978)

If the edges of the complete graph on $\mathbb{N}$ are colored with finitely many colors then the vertices can be ***covered by disjoint monochromatic paths of different color****.*

P. Erdős on Richard Rado:

"I was good at discovering perhaps difficult and interesting special cases, and Richard was good at generalizing them and putting them in their proper perspective."

Theorem (R. Rado, 1978)

*If the edges of the complete graph on $\mathbb{N}$ are colored with finitely many colors then the vertices can be **covered by disjoint monochromatic paths of different color**.*

P. Erdős on Richard Rado:

"I was good at discovering perhaps difficult and interesting special cases, and Richard was good at generalizing them and putting them in their proper perspective."

Theorem (R. Rado, 1978)

If the edges of the complete graph on $\mathbb{N}$ are colored with finitely many colors then the vertices can be ***covered by disjoint monochromatic paths of different color****.*

P. Erdős on Richard Rado:

"I was good at discovering perhaps difficult and interesting special cases, and Richard was good at generalizing them and putting them in their proper perspective."

Ideas of the proof

There is a non-trivial **0-1-valued measure** on $\mathbb{N}$, i.e. $m : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that:

- m is **finitely additive**,
- $m(\mathbb{N}) = 1$ **and** $m(\{n\}) = 0$ for all $n \in \mathbb{N}$.

Fact

- *If* $m(U \cup V) = 1$ *then either* $m(U) = 1$ *or* $m(V) = 1$.
- *If* $m(U) = m(V) = 1$ *then* $m(U \cap V) = 1$.

Ideas of the proof

There is a non-trivial **0-1-valued measure** on $\mathbb{N}$, i.e. $m : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that:

- m is **finitely additive**,
- $m(\mathbb{N}) = 1$ **and** $m(\{n\}) = 0$ for all $n \in \mathbb{N}$.

Fact

- *If* $m(U \cup V) = 1$ *then either* $m(U) = 1$ *or* $m(V) = 1$.
- *If* $m(U) = m(V) = 1$ *then* $m(U \cap V) = 1$.

Ideas of the proof

There is a non-trivial **0-1-valued measure** on $\mathbb{N}$, i.e. $m : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that:

- m is **finitely additive**,
- $m(\mathbb{N}) = 1$ **and** $m(\{n\}) = 0$ for all $n \in \mathbb{N}$.

Fact

- *If $m(U \cup V) = 1$ then either $m(U) = 1$ or $m(V) = 1$.*
- *If $m(U) = m(V) = 1$ then $m(U \cap V) = 1$.*

Ideas of the proof

There is a non-trivial **0-1-valued measure** on $\mathbb{N}$, i.e. $m : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that:

- m is **finitely additive**,
- $m(\mathbb{N}) = 1$ **and** $m(\{n\}) = 0$ for all $n \in \mathbb{N}$.

Fact

- *If $m(U \cup V) = 1$ then either $m(U) = 1$ or $m(V) = 1$.*
- *If $m(U) = m(V) = 1$ then $m(U \cap V) = 1$.*

There is a non-trivial **0-1-valued measure** on $\mathbb{N}$, i.e. $m : \mathcal{P}(\mathbb{N}) \to \{0,1\}$ such that:

- m is **finitely additive**,
- $m(\mathbb{N}) = 1$ **and** $m(\{n\}) = 0$ for all $n \in \mathbb{N}$.

Fact

- *If* $m(U \cup V) = 1$ *then either* $m(U) = 1$ *or* $m(V) = 1$.
- *If* $m(U) = m(V) = 1$ *then* $m(U \cap V) = 1$.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with red and blue edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is red$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is blue$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ **A_r is covered by a red path,**

$\Rightarrow$ repeat the **same for A_b simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with **red** and **blue** edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **red**$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **blue**$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ **A_r is covered by a red path,**

$\Rightarrow$ repeat the **same for A_b simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with **red** and **blue** edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **red**$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **blue**$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ **A_r is covered by a red path,**

$\Rightarrow$ repeat the **same for A_b simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with red and blue edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is red$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is blue$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ A_r **is covered by a red path,**

$\Rightarrow$ repeat the **same for A_b simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with red and blue edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is red$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is blue$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ A_r **is covered by a red path,**

$\Rightarrow$ repeat the **same for** A_b **simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with **red** and **blue** edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **red**$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **blue**$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ A_r **is covered by a red path,**

$\Rightarrow$ repeat the **same for** A_b **simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with **red** and **blue** edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ **is red**$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ **is blue**$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ A_r **is covered by a red path,**

$\Rightarrow$ repeat the **same for** A_b **simultaneously**.

Ideas of the proof

$m(U \cup V) = 1 \Rightarrow m(U) = 1$ or $m(V) = 1$;
$m(U) = m(V) = 1 \Rightarrow m(U \cap V) = 1$.

Consider a complete graph on $\mathbb{N}$ with **red** and **blue** edges.

- let $A_r = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **red**$\}) = 1\}$,
- let $A_b = \{u \in \mathbb{N} : m(\{v \in \mathbb{N} : \{u, v\}$ is **blue**$\}) = 1\}$,
- note that $\mathbb{N} = A_r \cup A_b$.

Any $u, u' \in A_r$ are connected by infinitely many red paths (of length 2),

$\Rightarrow$ **A_r is covered by a red path,**

$\Rightarrow$ repeat the **same for A_b simultaneously**.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that r **is small**:

1. ("easy") Every 2-edge colored K_n can be **partitioned into 2** monochromatic paths of different color.
2. **[K. Heinrich, ??]** There are r-edge colored copies of K_n for $r \geq 3$ so that there is no partition into r paths of different color.
3. **[A. Pokrovskiy, 2013]** Every 3-edge colored K_n can be **partitioned into 3** monochromatic paths.

Completely open: $r = 4$ **or larger.**

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that r **is small**:

1. ("easy") Every 2-edge colored K_n can be **partitioned into 2** monochromatic paths of different color.
2. **[K. Heinrich, ??]** There are r-edge colored copies of K_n for $r \geq 3$ so that there is no partition into r paths of different color.
3. **[A. Pokrovskiy, 2013]** Every 3-edge colored K_n can be **partitioned into 3** monochromatic paths.

Completely open: $r = 4$ **or larger.**

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that r **is small**:

1. ("easy") Every 2-edge colored K_n can be **partitioned into 2** monochromatic paths of different color.
2. **[K. Heinrich, ??]** There are r-edge colored copies of K_n for $r \geq 3$ so that there is no partition into r paths of different color.
3. **[A. Pokrovskiy, 2013]** Every 3-edge colored K_n can be **partitioned into 3** monochromatic paths.

Completely open: $r = 4$ or larger.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that r **is small**:

1. ("easy") Every 2-edge colored K_n can be **partitioned into 2** monochromatic paths of different color.
2. **[K. Heinrich, ??]** There are r-edge colored copies of K_n for $r \geq 3$ so that there is no partition into r paths of different color.
3. **[A. Pokrovskiy, 2013]** Every 3-edge colored K_n can be **partitioned into 3** monochromatic paths.

Completely open: $r = 4$ or larger.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that r **is small**:

1. ("easy") Every 2-edge colored K_n can be **partitioned into 2** monochromatic paths of different color.
2. **[K. Heinrich, ??]** There are r-edge colored copies of K_n for $r \geq 3$ so that there is no partition into r paths of different color.
3. **[A. Pokrovskiy, 2013]** Every 3-edge colored K_n can be **partitioned into 3** monochromatic paths.

Completely open: $r = 4$ or larger.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that r **is small**:

1. ("easy") Every 2-edge colored K_n can be **partitioned into 2** monochromatic paths of different color.
2. **[K. Heinrich, ??]** There are r-edge colored copies of K_n for $r \geq 3$ so that there is no partition into r paths of different color.
3. **[A. Pokrovskiy, 2013]** Every 3-edge colored K_n can be **partitioned into 3** monochromatic paths.

Completely open: $r = 4$ or larger.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

For arbitrary number of colors:

1. **[Gyárfás, 1989]** Every r-edge colored K_n is **covered by** $\leq C \cdot r^4$ monochromatic paths (for some small constant C).
2. **[Gyárfás et al., 1998]** Every r-edge colored copy of K_n can be **partitioned into** $\approx 100r\log(r)$ monochromatic cycles.

Significant work done on **monochromatic cycle partitions**; **Lehel's conjecture** and **Erdős-Gyárfás-Pyber conjecture**.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

For arbitrary number of colors:

1. **[Gyárfás, 1989]** Every r-edge colored K_n is **covered by** $\leq C \cdot r^4$ monochromatic paths (for some small constant C).
2. **[Gyárfás et al., 1998]** Every r-edge colored copy of K_n can be **partitioned into** $\approx 100r\log(r)$ monochromatic cycles.

Significant work done on **monochromatic cycle partitions**; **Lehel's conjecture** and **Erdős-Gyárfás-Pyber conjecture**.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

For arbitrary number of colors:

1. **[Gyárfás, 1989]** Every r-edge colored K_n is **covered by** $\leq C \cdot r^4$ monochromatic paths (for some small constant C).
2. **[Gyárfás et al., 1998]** Every r-edge colored copy of K_n can be **partitioned into** $\approx 100r\log(r)$ monochromatic cycles.

Significant work done on **monochromatic cycle partitions**; **Lehel's conjecture** and **Erdős-Gyárfás-Pyber conjecture**.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

For arbitrary number of colors:

1. **[Gyárfás, 1989]** Every r-edge colored K_n is **covered by** $\leq C \cdot r^4$ monochromatic paths (for some small constant C).
2. **[Gyárfás et al., 1998]** Every r-edge colored copy of K_n can be **partitioned into** $\approx 100r\log(r)$ monochromatic cycles.

Significant work done on **monochromatic cycle partitions**; **Lehel's conjecture** and **Erdős-Gyárfás-Pyber conjecture**.

Developments on the finite case

General problem (Gyárfás): given an r-edge coloring of K_n is there a cover by (disjoint) monochromatic paths (of different color)?

For arbitrary number of colors:

1. **[Gyárfás, 1989]** Every r-edge colored K_n is **covered by** $\leq C \cdot r^4$ monochromatic paths (for some small constant C).
2. **[Gyárfás et al., 1998]** Every r-edge colored copy of K_n can be **partitioned into** $\approx 100r\log(r)$ monochromatic cycles.

Significant work done on **monochromatic cycle partitions**; **Lehel's conjecture** and **Erdős-Gyárfás-Pyber conjecture**.

Definition

*Suppose that G is a graph and $k \in \mathbb{N}$. The k^{th} **power of G** is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a **finite path of length $\leq k$ from v to w**.*

What is a **power of a path**?

Definition

Suppose that G is a graph and $k \in \mathbb{N}$. The ***k^{th} power of G*** *is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a* ***finite path of length $\leq k$ from v to w****.*

What is a **power of a path**?

Definition

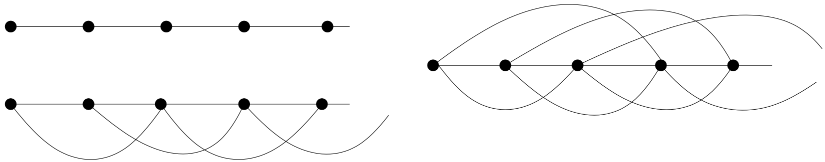
*Suppose that G is a graph and $k \in \mathbb{N}$. The **k^{th} power of G** is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a **finite path of length $\leq k$ from v to w**.*

What is a **power of a path**?

Definition

*Suppose that G is a graph and $k \in \mathbb{N}$. The k^{th} **power of G** is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a **finite path of length $\leq k$ from v to w**.*

What is a **power of a path**?



I.e. the graph is locally complete.

Motivation

Theorem (Infinite Ramsey)

*In every finite edge colored complete graph on $\mathbb{N}$ **there is an infinite monochromatic complete subgraph.***

- one **cannot always partition** into monochromatic complete subgraphs,
- how about partitions into **monochromatic locally complete subgraphs**?

Theorem (Infinite Ramsey)

In every finite edge colored complete graph on $\mathbb{N}$ ***there is an infinite monochromatic complete subgraph.***

- one cannot always partition into monochromatic complete subgraphs,
- how about partitions into monochromatic locally complete subgraphs?

Motivation

Theorem (Infinite Ramsey)

*In every finite edge colored complete graph on $\mathbb{N}$ **there is an infinite monochromatic complete subgraph.***

- one **cannot always partition** into monochromatic complete subgraphs,
- how about partitions into **monochromatic locally complete subgraphs**?

Motivation

Theorem (Infinite Ramsey)

In every finite edge colored complete graph on $\mathbb{N}$ ***there is an infinite monochromatic complete subgraph.***

- one **cannot always partition** into monochromatic complete subgraphs,
- how about partitions into **monochromatic locally complete subgraphs**?

Partitions into powers of paths

A **k^{th}-power of a path** is $\{x_i : i < n\}$ so that x_i, x_j is an edge if $|i-j| \leq k$.

Jointly with M. Elekes, L. Soukup and Z. Szentmiklóssy at Rényi Institute:

Theorem

*Fix natural numbers k, r and an r-edge coloring of the complete graph on $\mathbb{N}$. Then the vertices can be **covered by $\leq r^{(k-1)r+1}$ disjoint infinite monochromatic k^{th} powers of paths** apart from a finite set.*

For $k = r = 2$ we actually have a **partition into 4** monochromatic second powers of paths and this result is sharp.

Partitions into powers of paths

A k^{th}**-power of a path** is $\{x_i : i < n\}$ so that x_i, x_j is an edge if $|i - j| \leq k$.

Jointly with M. Elekes, L. Soukup and Z. Szentmiklóssy at Rényi Institute:

Theorem

Fix natural numbers k, r and an r-edge coloring of the complete graph on $\mathbb{N}$. Then the vertices can be ***covered by*** $\leq r^{(k-1)r+1}$ ***disjoint infinite monochromatic*** k^{th} ***powers of paths*** *apart from a finite set.*

For $k = r = 2$ we actually have a **partition into 4** monochromatic second powers of paths and this result is sharp.

Partitions into powers of paths

A **k^{th}-power of a path** is $\{x_i : i < n\}$ so that x_i, x_j is an edge if $|i - j| \leq k$.

Jointly with M. Elekes, L. Soukup and Z. Szentmiklóssy at Rényi Institute:

Theorem

Fix natural numbers k, r and an r-edge coloring of the complete graph on $\mathbb{N}$. Then the vertices can be ***covered by $\leq r^{(k-1)r+1}$ disjoint infinite monochromatic k^{th} powers of paths*** *apart from a finite set.*

For $k = r = 2$ we actually have a **partition into 4** monochromatic second powers of paths and this result is sharp.

Partitions into powers of paths

A k^{th}**-power of a path** is $\{x_i : i < n\}$ so that x_i, x_j is an edge if $|i - j| \leq k$.

Jointly with M. Elekes, L. Soukup and Z. Szentmiklóssy at Rényi Institute:

Theorem

Fix natural numbers k, r and an r-edge coloring of the complete graph on $\mathbb{N}$. Then the vertices can be ***covered by*** $\leq r^{(k-1)r+1}$ ***disjoint infinite monochromatic*** k^{th} ***powers of paths*** *apart from a finite set.*

For $k = r = 2$ we actually have a **partition into 4** monochromatic second powers of paths and this result is sharp.

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path,**
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on** $\mathbb{N}$ from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each** W_i,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

The tools of our proof

- introduce a **game** on edge colored graphs with parameter W (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
  - **a winning strategy for Bob covers W by a power of a path**,
  - find sufficient conditions on W for the existence of a winning strategy,
- use the **measure on $\mathbb{N}$** from before to find $\mathbb{N} = \bigcup\{W_i : i < N\}$ **with winning strategies on each W_i**,
- let Bob win **simultaneously** on each W_i.

Open problem: what is the **precise bound?** what about **finite graphs?**

Infinite paths of arbitrary length

Definition (Rado, 1978)

For a graph $P = (V, E)$, we say that P is a ***path*** iff there is a **well ordering** $\prec$ **on** V such that any two points $v, w \in V$ are connected by a $\prec$-monotone finite path.

How to imagine paths longer than the type of $\mathbb{N}$? **Type** $\mathbb{N} + 1$**?**

Infinite paths of arbitrary length

Definition (Rado, 1978)

For a graph $P = (V, E)$, we say that P is a ***path*** iff there is a **well ordering $\prec$ on V** such that any two points $v, w \in V$ are connected by a $\prec$-monotone finite path.

How to imagine paths longer than the type of $\mathbb{N}$? **Type $\mathbb{N} + 1$?**

Infinite paths of arbitrary length

Definition (Rado, 1978)

For a graph $P = (V, E)$, we say that P is a ***path*** iff there is a **well ordering $\prec$ on V** such that any two points $v, w \in V$ are connected by a $\prec$-monotone finite path.

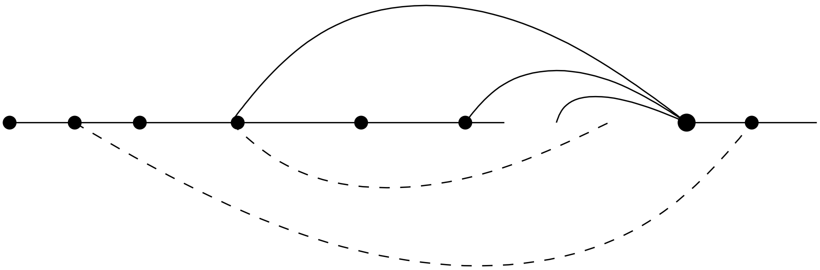
How to imagine paths longer than the type of $\mathbb{N}$? **Type $\mathbb{N} + 1$?**

Infinite paths of arbitrary length

Definition (Rado, 1978)

For a graph $P = (V, E)$, we say that P is a ***path*** iff there is a **well ordering $\prec$ on V** such that any two points $v, w \in V$ are connected by a $\prec$-monotone finite path.

How to imagine paths longer than the type of $\mathbb{N}$? **Type $\mathbb{N}+1$?**



Arbitrary infinite complete graphs

A **path** is a graph P with w.o. $\prec$ so that any two points are connected by a finite $\prec$-monotone path.

Problem (Rado, 1978)

Is every 2-edge colored infinite complete graph covered by two disjoint monochromatic paths of different color?

- How about **more colors**?

Arbitrary infinite complete graphs

A **path** is a graph P with w.o. $\prec$ so that any two points are connected by a finite $\prec$-monotone path.

Problem (Rado, 1978)

*Is every **2-edge colored** infinite complete graph covered by **two** disjoint monochromatic paths of different color?*

- How about **more colors**?

Arbitrary infinite complete graphs

A **path** is a graph P with w.o. $\prec$ so that any two points are connected by a finite $\prec$-monotone path.

Problem (Rado, 1978)

Is every ***2-edge colored*** *infinite complete graph covered by* ***two*** *disjoint monochromatic paths of different color?*

- How about more colors?

Arbitrary infinite complete graphs

Answers

Theorem (D.S.)

Every ***2-edge*** *colored infinite complete graph can be covered by* ***two disjoint monochromatic paths of different color****.*

Open: Is it true that the number of paths needed only **depends on the number of colors**? The number of colors and the **size** of the graph?

Theorem (D.S.)

*Every **2-edge** colored infinite complete graph can be covered by **two disjoint monochromatic paths of different color**.*

Open: Is it true that the number of paths needed only depends on the number of colors? The number of colors and the size of the graph?

Theorem (D.S.)

*Every **finite edge** colored infinite complete graph can be covered by **finitely many disjoint monochromatic paths**.*

Open: Is it true that the number of paths needed only **depends on the number of colors**? The number of colors and the **size** of the graph?

Theorem (D.S.)

*Every **finite edge** colored infinite complete graph can be covered by **finitely many disjoint monochromatic paths**.*

Open: Is it true that the number of paths needed only **depends on the number of colors**? The number of colors and the **size** of the graph?

How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the limit points of the path first,
2. find the cofinal sets witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

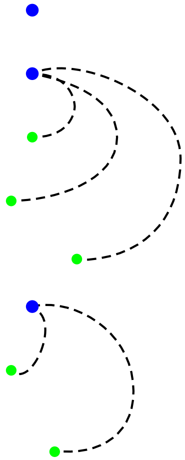
1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

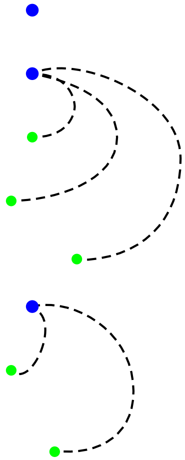


How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

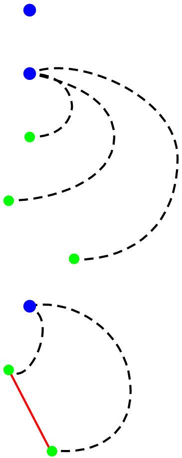


How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

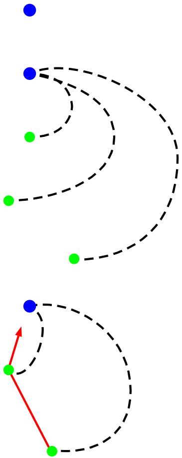


How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

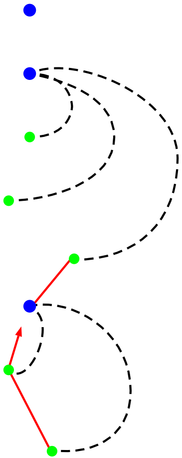


How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

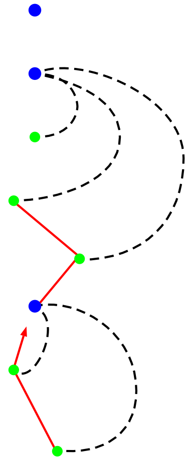


How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…

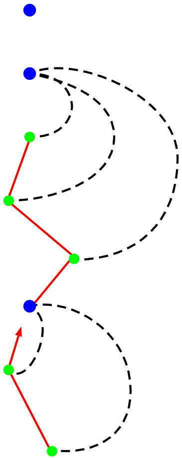


How to build uncountable paths?

... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**...

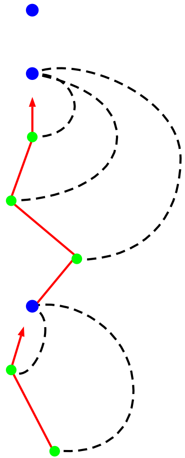


How to build uncountable paths?

… and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,
2. find the **cofinal sets** witnessing the limit position,
3. provided that the green vertices are connected (by finite paths), **build the transfinite path around these points**…



Vertex coloring problems

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*

How does **large chromatic number** affect the **subgraph structure**?

Vertex coloring problems

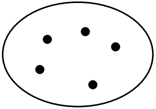
Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*

How does large chromatic number affect the subgraph structure?

Definition

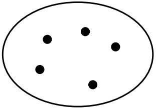
*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*

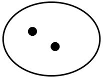


How does large chromatic number affect the subgraph structure?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*



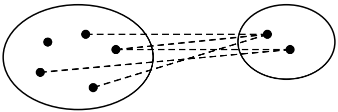


How does large chromatic number affect the subgraph structure?

Vertex coloring problems

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*

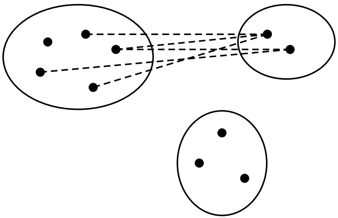


How does large chromatic number affect the subgraph structure?

Vertex coloring problems

Definition

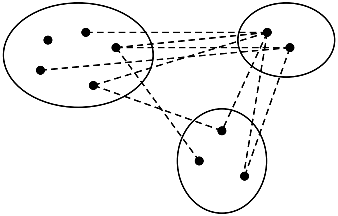
The ***chromatic number*** *of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that* ***the vertices of G can be covered by κ many independent sets****.*



How does **large chromatic number** affect the **subgraph structure**?

Definition

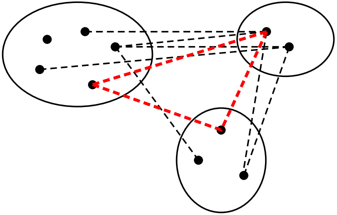
The ***chromatic number*** *of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that* ***the vertices of G can be covered by κ many independent sets****.*



How does large chromatic number affect the subgraph structure?

Definition

*The **chromatic number** of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that **the vertices of G can be covered by κ many independent sets**.*

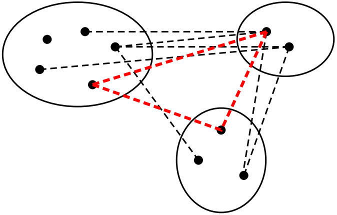


How does large chromatic number affect the subgraph structure?

Vertex coloring problems

Definition

The ***chromatic number*** *of a graph G, denoted by $Chr(G)$, is the least (cardinal) number κ such that* ***the vertices of G can be covered by κ many independent sets****.*



How does **large chromatic number** affect the **subgraph structure**?

The first results

- **Tutte, 1954:** There are $\triangle$-free graphs of **arbitrary large finite chromatic number**.
- **Erdős, 1959:** There are graphs with **arbitrary large girth** and **arbitrary large finite chromatic number**.
  - **the probability method**;
  - de-randomized constructions: **expanders**.

The first results

- **Tutte, 1954:** There are △-**free** graphs of **arbitrary large finite chromatic number**.

- **Erdős, 1959:** There are graphs with **arbitrary large girth** and **arbitrary large finite chromatic number**.
  - the probability method;
  - de-randomized constructions: expanders.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

The first results

- **Tutte, 1954:** There are △-free graphs of arbitrary large finite chromatic number.
- **Erdős, 1959:** There are graphs with arbitrary large girth and arbitrary large finite chromatic number.
    - the probability method;
    - de-randomized constructions: expanders.

Photograph by J. A. Bondy.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

The first results

- **Tutte, 1954:** There are △-free graphs of **arbitrary large finite chromatic number**.

- **Erdős, 1959:** There are graphs with **arbitrary large girth** and **arbitrary large finite chromatic number**.
  - **the probability method**;
  - de-randomized constructions: **expanders**.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

The first results

- **Tutte, 1954:** There are $\triangle$-**free** graphs of **arbitrary large finite chromatic number**.

- **Erdős, 1959:** There are graphs with **arbitrary large girth** and **arbitrary large finite chromatic number**.
  - **the probability method**;
  - de-randomized constructions: **expanders**.

Photograph by J. A. Bondy.

Two giants of combinatorics share a passion: Erdős and William T. Tutte play "Go" at Tutte's home in Westmontrose, Ontario, 1985. Another favorite game of Erdős's was Ping-Pong.

Uncountable chromatic number

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are △-**free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

In particular, cyles of length 4 embed into G.

Uncountable chromatic number

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are $\triangle$-**free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.

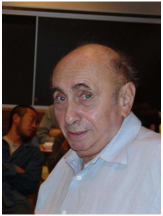
In particular, cyles of length 4 embed into G.

Uncountable chromatic number

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are $\triangle$-**free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.



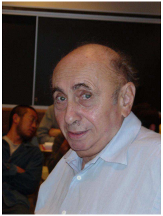
In particular, cyles of length 4 embed into G.

Uncountable chromatic number

What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959:** There are $\triangle$-**free** graphs with size and **chromatic number** κ for each infinite κ.

- **Erdős-Hajnal, 1966:** If $Chr(G) > \omega$ then K_{n,ω_1} **embeds** into G for each $n \in \omega$.



In particular, **cyles of length 4** embed into G.

Degrees of connectivity

Definition

*G is n-connected (**infinitely connected**) iff given vertices v, w and $n-1$ points (**finitely many points**) F there is a path which connects v and w and avoids F.*

E.g: K_{n,ω_1} is n-connected. **Does having large chromatic number**

- imply the existence of **highly connected** subgraphs?
- **reflect** to some highly connected subgraphs?

Note: the chromatic number **reflects to connected subgraphs**.

Degrees of connectivity

Definition

*G is **n-connected** (**infinitely connected**) iff given vertices v, w and $n-1$ **points** (**finitely many points**) F there is a path which connects v and w and avoids F.*

E.g: K_{n,ω_1} is n-connected. **Does having large chromatic number**

- imply the existence of **highly connected** subgraphs?
- **reflect** to some highly connected subgraphs?

Note: the chromatic number **reflects to connected subgraphs**.

Definition

*G is **n-connected** (**infinitely connected**) iff given vertices v, w and **n − 1 points** (**finitely many points**) F there is a path which connects v and w and avoids F.*

E.g: K_{n,ω_1} is n-connected. Does having large chromatic number

- imply the existence of **highly connected** subgraphs?
- **reflect** to some highly connected subgraphs?

Note: the chromatic number **reflects to connected subgraphs**.

Degrees of connectivity

Definition

G is n-connected (infinitely connected) iff given vertices v, w and $n-1$ points (finitely many points) F there is a path which connects v and w and avoids F.

E.g: K_{n,ω_1} is n-connected. **Does having large chromatic number**

- imply the existence of **highly connected** subgraphs?
- **reflect** to some highly connected subgraphs?

Note: the chromatic number **reflects to connected subgraphs**.

Degrees of connectivity

Definition

*G is n-connected (**infinitely connected**) iff given vertices v, w and $n-1$ **points** (**finitely many points**) F there is a path which connects v and w and avoids F.*

E.g: K_{n,ω_1} is n-connected. **Does having large chromatic number**

- imply the existence of **highly connected** subgraphs?
- reflect to some highly connected subgraphs?

Note: the chromatic number reflects to connected subgraphs.

Degrees of connectivity

Definition

*G is n-connected (**infinitely connected**) iff given vertices v, w and n − 1 points (**finitely many points**) F there is a path which connects v and w and avoids F.*

E.g: K_{n,ω_1} is n-connected. **Does having large chromatic number**

- imply the existence of **highly connected** subgraphs?
- **reflect** to some highly connected subgraphs?

Note: the chromatic number **reflects to connected subgraphs**.

Degrees of connectivity

Definition

*G is **n-connected** (**infinitely connected**) iff given vertices v, w and **n − 1 points** (**finitely many points**) F there is a path which connects v and w and avoids F.*

E.g: K_{n,ω_1} is n-connected. **Does having large chromatic number**

- imply the existence of **highly connected** subgraphs?
- **reflect** to some highly connected subgraphs?

Note: the chromatic number **reflects to connected subgraphs**.

Chromatic number and connectivity

Does every graph G with uncountable chromatic number contain

- an *n*-**connected uncountably chromatic** subgraph?
- an **infinitely connected uncountably chromatic** subgraph?
  - **[Erdős-Hajnal, 1966]** Suppose $|G| = Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?
  - **[Erdős-Hajnal, 1985]** Suppose $Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?

Chromatic number and connectivity

Does every graph G with uncountable chromatic number contain

- an n-**connected uncountably chromatic** subgraph?
- an **infinitely connected uncountably chromatic** subgraph?
  - **[Erdős-Hajnal, 1966]** Suppose $|G| = Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?
  - **[Erdős-Hajnal, 1985]** Suppose $Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?

Chromatic number and connectivity

Does every graph G with uncountable chromatic number contain

- an ***n*-connected uncountably chromatic** subgraph?
- an **infinitely connected uncountably chromatic** subgraph?
  - **[Erdős-Hajnal, 1966]** Suppose $|G| = Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?
  - **[Erdős-Hajnal, 1985]** Suppose $Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?

Chromatic number and connectivity

Does every graph G with uncountable chromatic number contain

- an *n-connected uncountably chromatic* subgraph?
- an **infinitely connected uncountably chromatic** subgraph?
  - **[Erdős-Hajnal, 1966]** Suppose $|G| = Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?
  - **[Erdős-Hajnal, 1985]** Suppose $Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?

Chromatic number and connectivity

Does every graph G with uncountable chromatic number contain

- an **n-connected uncountably chromatic** subgraph?
- an **infinitely connected uncountably chromatic** subgraph?
  - **[Erdős-Hajnal, 1966]** Suppose $|G| = Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?
  - **[Erdős-Hajnal, 1985]** Suppose $Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?

Chromatic number and connectivity

Does every graph G with uncountable chromatic number contain

- an n-**connected uncountably chromatic** subgraph?
- an **infinitely connected uncountably chromatic** subgraph?
  - **[Erdős-Hajnal, 1966]** Suppose $|G| = Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?
  - **[Erdős-Hajnal, 1985]** Suppose $Chr(G) = \omega_1$. Is there an infinitely connected uncountably chromatic subgraph of G?

The answers

- **[Komjáth, 1986]**
  If $Chr(G) > \omega$ then there is an
  ***n*-connected** uncountably
  chromatic subgraph of G
  for each $n \in \omega$.

- **[Komjáth, 1988]** Under PFA, if $|G| = Chr(G) = \omega_1$ then
  G **has an infinitely connected** uncountably chromatic subgraph.
- **[Komjáth, 1988-2013]** Consistently, there is a graph
  $|G| = Chr(G) = \omega_1$ such that there is **no infinitely connected**
  uncountably chromatic subgraphs of G.

The answers

- **[Komjáth, 1986]**
  If $Chr(G) > \omega$ then there is an **n-connected** uncountably chromatic subgraph of G for each $n \in \omega$.

- **[Komjáth, 1988]** Under PFA, if $|G| = Chr(G) = \omega_1$ then G **has an infinitely connected** uncountably chromatic subgraph.
- **[Komjáth, 1988-2013]** Consistently, there is a graph $|G| = Chr(G) = \omega_1$ such that there is **no infinitely connected** uncountably chromatic subgraphs of G.

The answers

- **[Komjáth, 1986]**
  If $Chr(G) > \omega$ then there is an **n-connected** uncountably chromatic subgraph of G for each $n \in \omega$.

- **[Komjáth, 1988]** Under PFA, if $|G| = Chr(G) = \omega_1$ then **G has an infinitely connected** uncountably chromatic subgraph.
- **[Komjáth, 1988-2013]** Consistently, there is a graph $|G| = Chr(G) = \omega_1$ such that there is **no infinitely connected** uncountably chromatic subgraphs of G.

The answers

- **[Komjáth, 1986]**
  If $Chr(G) > \omega$ then there is an *n*-**connected** uncountably chromatic subgraph of G for each $n \in \omega$.

- **[Komjáth, 1988]** Under PFA, if $|G| = Chr(G) = \omega_1$ then G **has an infinitely connected** uncountably chromatic subgraph.
- **[Komjáth, 1988-2013]** Consistently, there is a graph $|G| = Chr(G) = \omega_1$ such that there is **no infinitely connected** uncountably chromatic subgraphs of G.

The answers - continued

- Question from [Erdős-Hajnal, 1966]: independent. ✓
- Question from [Erdős-Hajnal, 1985]: consistently no.

Theorem (D.S. 2014)

*There is a graph of **chromatic number** ω_1 and size continuum **without uncountable infinitely connected subgraphs**.*

- Question from [Erdős-Hajnal, 1985]: no in ZFC. ✓

The answers - continued

- Question from **[Erdős-Hajnal, 1966]**: **independent**. ✓
- Question from **[Erdős-Hajnal, 1985]**: **consistently no**.

Theorem (D.S. 2014)

There is a graph of ***chromatic number*** ω_1 *and size continuum* ***without uncountable infinitely connected subgraphs****.*

- Question from **[Erdős-Hajnal, 1985]**: **no in ZFC**. ✓

The answers - continued

- Question from **[Erdős-Hajnal, 1966]**: **independent**. ✓
- Question from **[Erdős-Hajnal, 1985]**: **consistently no**.

Theorem (D.S. 2014)

There is a graph of ***chromatic number*** ω_1 *and size continuum* ***without uncountable infinitely connected subgraphs****.*

- Question from **[Erdős-Hajnal, 1985]**: **no in ZFC**. ✓

The answers - continued

- Question from **[Erdős-Hajnal, 1966]**: **independent**. ✓
- Question from **[Erdős-Hajnal, 1985]**: **consistently no**.

Theorem (D.S. 2014)

*There is a graph of **chromatic number** ω_1 and size continuum **without uncountable infinitely connected subgraphs**.*

- Question from **[Erdős-Hajnal, 1985]**: **no in ZFC**. ✓

The answers - continued

- Question from **[Erdős-Hajnal, 1966]**: **independent**. ✓
- Question from **[Erdős-Hajnal, 1985]**: **consistently no**.

Theorem (D.S. 2014)

There is a graph of ***chromatic number*** ω_1 *and size continuum* ***without uncountable infinitely connected subgraphs****.*

- Question from **[Erdős-Hajnal, 1985]**: **no in ZFC**. ✓

A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.

A few words about the proof

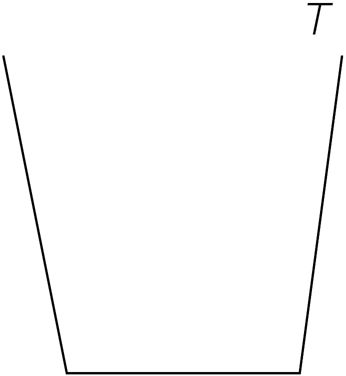
- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.

A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- thin out the edges to have no uncountable infinitely connected subgraph:
  - use a ladder system on the tree;
- how to make sure that the chromatic number is still large?
  - a smart diagonalization of length continuum.

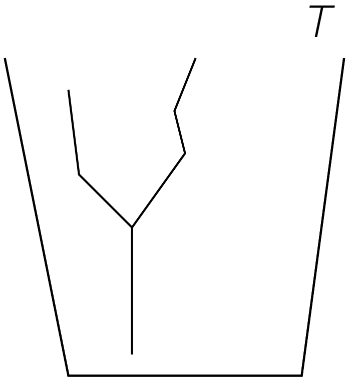
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



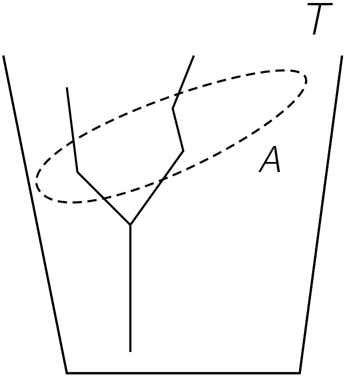
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



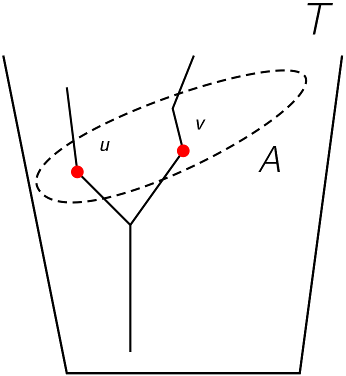
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



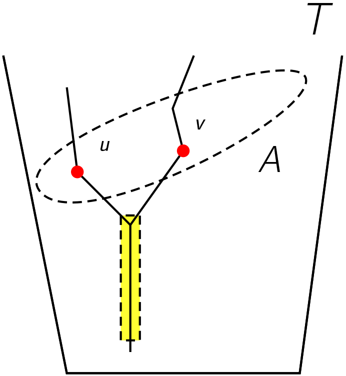
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



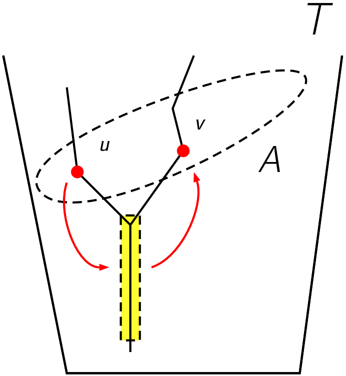
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



A few words about the proof

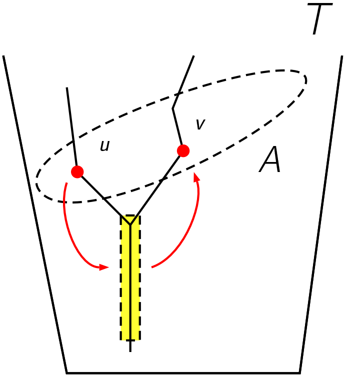
- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;

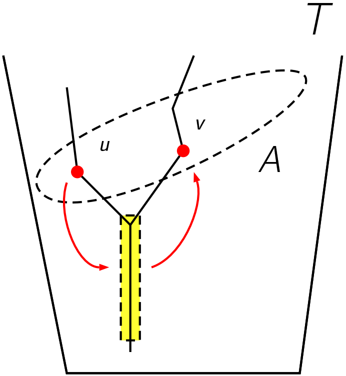
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;



- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.

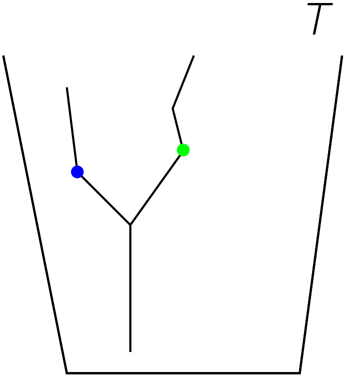
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



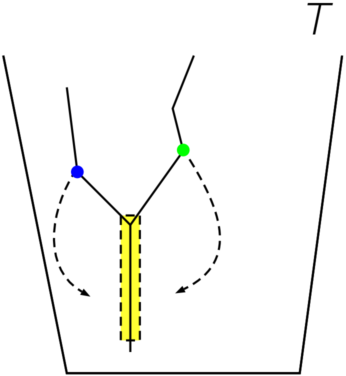
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



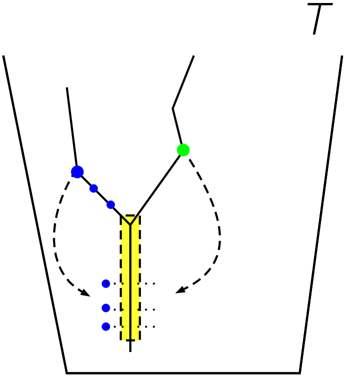
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



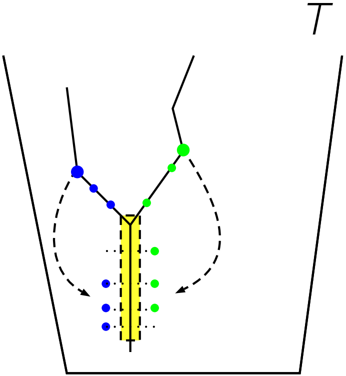
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum.**



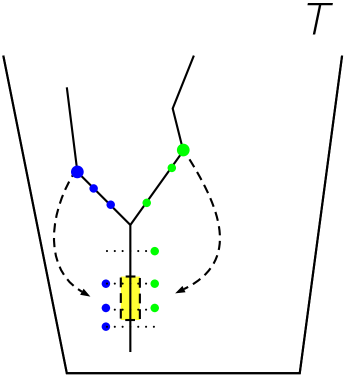
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



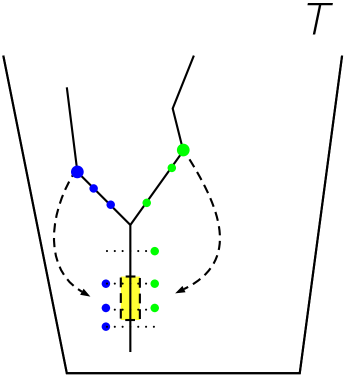
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



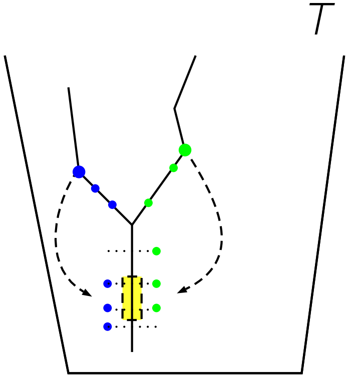
A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



A few words about the proof

- find a rather **disconnected graph with large chromatic number**:
  - the comparability graph of a **non-special tree** without uncountable chains;
- **thin out the edges** to have no uncountable infinitely connected subgraph:
  - use a **ladder system** on the tree;
- how to make sure that **the chromatic number is still large?**
  - a **smart diagonalization of length continuum**.



Problems I would like to solve I.

... or see solved.

- [Komjáth, ??] Is there a graph with uncountable chromatic number which contains no infinitely connected subgraphs?
  - Recall: there are G with $Chr(G) > \omega$ where every infinitely connected subgraph is countable.
  - Question: how to eliminate the countable infinitely connected subgraphs?
  - Not even a consistency result...

Problems I would like to solve I.

... or see solved.

- **[Komjáth, ??]** Is there a graph with uncountable chromatic number which contains **no infinitely connected subgraphs**?
  - Recall: there are G with $Chr(G) > \omega$ where every infinitely connected subgraph is countable.
  - Question: **how to eliminate** the countable infinitely connected subgraphs?
  - **Not even a consistency result...**

Problems I would like to solve I.

... or see solved.

- **[Komjáth, ??]** Is there a graph with uncountable chromatic number which contains **no infinitely connected subgraphs**?
  - Recall: there are G with $Chr(G) > \omega$ where every infinitely connected subgraph is countable.
  - Question: **how to eliminate** the countable infinitely connected subgraphs?
  - **Not even a consistency result...**

Problems I would like to solve I.

... or see solved.

- **[Komjáth, ??]** Is there a graph with uncountable chromatic number which contains **no infinitely connected subgraphs**?
  - Recall: there are G with $Chr(G) > \omega$ where every infinitely connected subgraph is countable.
  - Question: **how to eliminate** the countable infinitely connected subgraphs?
  - **Not even a consistency result...**

Problems I would like to solve I.

... or see solved.

- **[Komjáth, ??]** Is there a graph with uncountable chromatic number which contains **no infinitely connected subgraphs**?
  - Recall: there are G with $Chr(G) > \omega$ where every infinitely connected subgraph is countable.
  - Question: **how to eliminate** the countable infinitely connected subgraphs?
  - **Not even a consistency result...**

Problems I would like to solve II.

... or see solved.

- **[Erdős, Hajnal 1975]** Is there a graph with uncountable chromatic number which contains **no triangle free subgraphs with uncountable chromatic number**?
  - Recall: $\exists$ Δ-free graph G with $Chr(G) > \omega$.
  - Question: is there a graph with large chromatic number **without such a subgraph**?
  - **[Shelah, 1988] Consistently yes.**

Problems I would like to solve II.

... or see solved.

- **[Erdős, Hajnal 1975]** Is there a graph with uncountable chromatic number which contains **no triangle free subgraphs with uncountable chromatic number**?
  - Recall: $\exists$ **Δ-free graph G with** $Chr(G) > \omega$.
  - Question: is there a graph with large chromatic number **without such a subgraph**?
  - **[Shelah, 1988] Consistently yes.**

Problems I would like to solve II.

... or see solved.

- **[Erdős, Hajnal 1975]** Is there a graph with uncountable chromatic number which contains **no triangle free subgraphs with uncountable chromatic number**?
  - Recall: $\exists$ **Δ-free graph G with $Chr(G) > \omega$.**
  - Question: is there a graph with large chromatic number **without such a subgraph**?
  - **[Shelah, 1988] Consistently yes.**

Problems I would like to solve II.

... or see solved.

- **[Erdős, Hajnal 1975]** Is there a graph with uncountable chromatic number which contains **no triangle free subgraphs with uncountable chromatic number**?
  - Recall: $\exists$ **Δ-free graph** G **with** $Chr(G) > \omega$.
  - Question: is there a graph with large chromatic number **without such a subgraph**?
  - **[Shelah, 1988] Consistently yes.**

Problems I would like to solve II.

... or see solved.

- **[Erdős, Hajnal 1975]** Is there a graph with uncountable chromatic number which contains **no triangle free subgraphs with uncountable chromatic number**?
  - Recall: $\exists$ **Δ-free graph G with $Chr(G) > \omega$.**
  - Question: is there a graph with large chromatic number **without such a subgraph**?
  - **[Shelah, 1988] Consistently yes.**

Problems I would like to solve III.

... or see solved.

- **[Erdős, Hajnal 1975]** Suppose that $f : \mathbb{N} \to \mathbb{N}$ is increasing. Is there a graph G with uncountable chromatic number such that **every n-chromatic subgraph of G has at least $f(n)$ vertices** (for all $n \geq 3$)?
  - Recall: If $Chr(G)$ is infinite then $\sup\{Chr(H) : H \subseteq G \text{ finite}\}$ is **infinite** as well.
  - Question: **how fast** are the finitely chromatic subgraphs **growing**?
  - **[Shelah, 2005] Consistently yes.**

Problems I would like to solve III.

... or see solved.

- **[Erdős, Hajnal 1975]** Suppose that $f : \mathbb{N} \to \mathbb{N}$ is increasing. Is there a graph G with uncountable chromatic number such that **every n-chromatic subgraph of G has at least $f(n)$ vertices** (for all $n \geq 3$)?
  - Recall: If $Chr(G)$ is infinite then $\sup\{Chr(H) : H \subseteq G$ **finite**$\}$ **is infinite** as well.
  - Question: **how fast** are the finitely chromatic subgraphs **growing**?
  - **[Shelah, 2005] Consistently yes.**

Problems I would like to solve III.

... or see solved.

- **[Erdős, Hajnal 1975]** Suppose that $f : \mathbb{N} \to \mathbb{N}$ is increasing. Is there a graph G with uncountable chromatic number such that **every n-chromatic subgraph of G has at least $f(n)$ vertices** (for all $n \geq 3$)?
  - Recall: If $Chr(G)$ is infinite then $\sup\{Chr(H) : H \subseteq G$ **finite**$\}$ **is infinite** as well.
  - Question: **how fast** are the finitely chromatic subgraphs **growing**?
  - **[Shelah, 2005] Consistently yes.**

Problems I would like to solve III.

... or see solved.

- **[Erdős, Hajnal 1975]** Suppose that $f : \mathbb{N} \to \mathbb{N}$ is increasing. Is there a graph G with uncountable chromatic number such that **every n-chromatic subgraph of G has at least $f(n)$ vertices** (for all $n \geq 3$)?
  - Recall: If $Chr(G)$ is infinite then $\sup\{Chr(H) : H \subseteq G$ **finite**$\}$ **is infinite** as well.
  - Question: **how fast** are the finitely chromatic subgraphs **growing**?
  - **[Shelah, 2005] Consistently yes.**

Problems I would like to solve III.

... or see solved.

- **[Erdős, Hajnal 1975]** Suppose that $f : \mathbb{N} \to \mathbb{N}$ is increasing. Is there a graph G with uncountable chromatic number such that **every n-chromatic subgraph of G has at least $f(n)$ vertices** (for all $n \geq 3$)?
  - Recall: If $Chr(G)$ is infinite then $\sup\{Chr(H) : H \subseteq G$ **finite**$\}$ **is infinite** as well.
  - Question: **how fast** are the finitely chromatic subgraphs **growing**?
  - **[Shelah, 2005] Consistently yes.**

Thank you for your attention.

"*The infinite we do now, the finite will have to wait a little.*"

P. Erdős