THE UNION OF TWO D-SPACES NEED NOT BE D

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Abstract. We construct from a version of $\diamondsuit$ a T_2 example of a hereditarily Lindelöf space X that is not a D-space but it is the union of two subspaces both of which are D-spaces. This answers a question of Arhangel'skii.

A T_1 space X is said to be a D*-space* if for each open neighbourhood assignment $\{U_x : x \in X\}$ there is a closed and discrete subset $D \subseteq X$ such that $\{U_x : x \in D\}$ covers the space. The notion is due to van Douwen and was first studied in [2]. The main open question regarding D-spaces is whether every regular Lindelöf space is a D-space. Recently in [4] the construction of a consistent T_2 counterexample to the van Douwen question was presented. In this note we use the same technique to construct an example of a T_2 space that is not a D-space but is the union of two subspaces that are both D-spaces. This answers a question of Arhangel'skii from [1].

A topology on ω_1 is defined by constructing a sequence $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of subsets of ω_1 such that $\alpha \in U_\alpha$. The example will be obtained by taking the family $\mathcal{U} \cup \{\omega_1 \setminus H : H \in [\omega_1]^{<\omega}\}$ as a subbasis. Then sets of the form $U_F \setminus H$ where $F, H \subseteq \omega_1$ are finite and $U_F = \bigcap_{\alpha \in F} U_\alpha$ form a basis for the topology. Any such topology is T_1 and there is a natural way to make it T_2 by identifying ω_1 in an appropriate way with some other T_2 space and taking the common refinement of the two topologies.

We also partition ω_1 as a union of two stationary sets $S_0 \cup S_1$. We will construct the U_α's in such a way that $\alpha \in U_\alpha$ is the neighborhood assignment witnessing the space is not D but both subspaces S_0 and S_1 are D-spaces. Whether the union of two D-spaces is always a D-space was asked in [1].

The following lemma shows how the subspaces will be made to be D.

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Lemma 0.1. *Suppose that τ is a topology on ω_1 obtained by taking a family $\{U_\alpha : \alpha \in \omega_1\} \cup \{\omega_1 \setminus F : F \in [\omega_1]^{<\omega}\}$ as a subbasis. Suppose that $S \subseteq \omega_1$ is an uncountable subspace. Suppose also that for any uncountable $T \subseteq S$ and any neighborhood assignment $\{V_\alpha : \alpha \in T\}$ such that each $V_\alpha = U_{F_\alpha}$ and the family $\{F_\alpha : \alpha \in T\}$ is pairwise disjoint, there is a $D \subseteq T$ countable and closed discrete in S such that $\{U_{F_\alpha} : \alpha \in D\}$ covers a tail of S. Then the subspace S is hereditarily a D-space.*

PROOF. Fix an arbitrary neighborhood assignment $\mathcal{V} = \{V_\alpha : \alpha \in S'\}$ with $S' \subseteq S$. Without loss of generality we may assume $V_\alpha = U_{F_\alpha} \setminus G_\alpha$ for some finite F_α and G_α. Let M be a countable elementary submodel of some $H(\kappa)$ for κ sufficiently large so that

$$\{V_\alpha, F_\alpha, G_\alpha : \alpha \in S'\} \in M.$$

Enumerate as $\{d_n : n \in \omega\}$ the finite subsets of $S' \cap M$. Also enumerate $S' \cap M = \{\beta_n : n \in \omega\}$. We define a sequence $\{E_n : n \in \omega\}$ as follows. First consider d_0.

If there is a $\gamma \in S'$ such that $F_\gamma = d_0$ by elementarity we may fix $\gamma_0 \in S' \cap M$ such that $F_{\gamma_0} = d_0$.

If there is an uncountable T_0 such that $\{F_\alpha : \alpha \in T_0\}$ is an uncountable Δ-system with root d_0, fix such a T_0 and consider the family $\{U_{F_\alpha \setminus d_0} : \alpha \in T_0\}$. By assumption there is a $D_0 \subseteq T_0$ countable and closed discrete in S such that $\{U_{F_\alpha \setminus d_0} : \alpha \in D_0\}$ covers a tail of S. By elementarity we may assume that $D_0 \in M$ and that

$$S \setminus M \subseteq \bigcup_{\alpha \in D_0} U_{F_\alpha \setminus d_0}.$$

If there is no such T_0 just let $D_0 = \emptyset$.

Finally let k_0 be minimal such that $\beta_{k_0} \notin \bigcup\{U_{F_\alpha} \setminus G_\alpha : \alpha \in D_0 \cup \{\gamma_0\}\}$. Now let $E_0 = \{\gamma_0\} \cup D_0 \cup \{\beta_{k_0}\}$.

Suppose $n > 0$ and we have constructed $E_0 \subseteq ... \subseteq E_{n-1}$ and $E_i \in M$ are countable and closed and discrete in S for each $i < n$. Let

$$S_n = S' \setminus \Big(\bigcup_{\alpha \in E_{n-1}} U_{F_\alpha} \setminus G_\alpha \Big).$$

And consider d_n.

If there is a $\gamma \in S_n$ such that $F_\beta = d_n$ then by elementarity we may fix $\gamma_n \in S_n \cap M$ with $F_{\gamma_n} = d_n$.

If there is an uncountable $T_n \subseteq S_n$ such that $\{F_\alpha : \alpha \in T_n\}$ is a Δ-system with root d_n, fix such a T_n. Proceed now as above, finding a countable $D_n \in M$ subset of T_n closed discrete in S such that $\{U_{F_\alpha} :$

$\alpha \in D_n\}$ covers $S \setminus M$. If there is no such uncountable T_n just let $D_n = \emptyset$. And finally let k_n be minimal such that

$$\beta_{k_n} \notin \bigcup \{U_{F_\alpha} \setminus G_\alpha : \alpha \in E_{n-1} \cup D_n \cup \{\gamma_{n+1}\}\ \}.$$

Finally let $E_n = E_{n-1} \cup D_n \cup \{\gamma_n, \beta_{k_n}\}$.
Now, let

$$D = \bigcup_{n \in \omega} E_n.$$

Claim 0.2. $S' \subseteq \bigcup \{U_{F_\alpha} \setminus G_\alpha : \alpha \in D\}$

PROOF. Clearly by choice of the β_{n_k} it must be the case that $S' \cap M$ is covered. So fix $\gamma \in S' \setminus M$.

First consider the possibility that $F_\gamma \subseteq M$. If so, then by elementarity, there is a $\beta \in S' \cap M$ such that $F_\beta = F_\gamma$. Fix n such that $d_n = F_\gamma$ and consider stage n of the construction. If $\gamma \notin \bigcup_{\alpha \in E_{n-1}} U_{F_\alpha} \setminus G_\alpha$, then at this stage we fixed γ_n with $F_{\gamma_n} = d_n$ and we put $\gamma_n \in E_n \subseteq D$. Then since $\gamma \in U_{F_\gamma}$ it follows that $\gamma \in U_{F_{\gamma_n}}$. And since $\gamma_n \in M$ it follows that $G_{\gamma_n} \subseteq M$. Therefore $\gamma \in U_{F_{\gamma_n}} \setminus G_{\gamma_n}$ as required since $\gamma_n \in D$.

Next, consider the possibility that $F_\gamma \setminus M \neq \emptyset$. Then there is an n such that $d_n = F_\gamma \cap M$. By the elementary submodel proof of the Δ-system lemma (see [3] or for an explicit proof see [4]) it follows that there is an uncountable Δ-system of the form $\{F_\alpha : \alpha \in T_n\}$ with root d_n where $T_n \subseteq S_n$. By choice of D_n we may fix $\alpha \in D_n$ such that $\gamma \in U_{F_\alpha \setminus d_n}$. And since $\gamma \in U_{d_n}$ it follows that $\gamma \in U_{F_\alpha}$. Finally since $\alpha \in M$ it follows that $G_\alpha \subseteq M$ so $\gamma \in U_{F_\alpha} \setminus G_\alpha$ as required since $\alpha \in D_n \subseteq D$. □

Claim 0.3. *D is closed discrete in S'.*

PROOF. This follows directly from the following observation: Suppose that X is a space, $\{V_x : x \in X\}$ a neighborhood assignment and $\{B_n : n \in \omega\}$ a family of closed discrete subsets such that

(1) $X = \bigcup \{V_x : x \in \bigcup_{k<\omega} B_k\}$, and
(2) $B_n \subseteq X \setminus \bigcup \{V_x : x \in \bigcup_{k<n} B_k\}$

Then $\bigcup_n B_n$ is closed discrete. □ □

Remark: If the family of sets $\{U_\alpha : \alpha \in \omega_1\}$ generates a Hausdorff topology, then the lemma still applies and the proof is in fact simplified since the extra parameter of the complement of the finite sets can be removed.

Now let us proceed with the construction of the example. The topology will be a common refinement of the topology generated by a sequence of subsets $U_\alpha \subseteq \omega_1$ and by identifying ω_1 with a subset of $[\mathbb{R}]^{<\omega}$ and using Euclidean open subsets to define a topology. In particular:

Definition 0.4. *Define a topology on $[\mathbb{R}]^{<\omega}$ as follows. Let $Q \subseteq \mathbb{R}$ be a Euclidean open set and let $Q^* = \{H \in [\mathbb{R}]^{<\omega} : H \subseteq Q\}$. Sets of the form Q^* define a topology ρ on $[\mathbb{R}]^{<\omega}$.*

The proof of the following claim is straightforward.

Claim 0.5. *(1) $([\mathbb{R}]^{<\omega}, \rho)$ is of countable weight,*
(2) any family $\mathcal{X} \subseteq [\mathbb{R}]^{<\omega}$ of pairwise disjoint nonempty sets forms a Hausdorff subspace of $([\mathbb{R}]^{<\omega}, \rho)$.

Let us fix a countable base $\mathcal{W}$ for $([\mathbb{R}]^{<\omega}, \rho)$.

To proceed with the rest of the construction we assume $\diamondsuit$ and fix two sequences:

(1) $\{C_\alpha : \alpha \in \omega_1\}$ an enumeration of $[\omega_1]^\omega$ such that $C_\alpha \subseteq \alpha$ for each α.
(2) $\{a_\alpha : \alpha \in \omega_1\}$ a special $\diamondsuit$ sequence that captures functions on S_0 stationarily often on S_1 and vice-versa in the following sense:
   (a) for each uncountable partial function $f : S_0 \to [\omega_1]^{<\omega}$ the set of $\alpha \in S_1$ such that $f \restriction \mathrm{dom}(f) \cap \alpha = a_\alpha$ is stationary, and
   (b) for each uncountable partial function $f : S_1 \to [\omega_1]^{<\omega}$ the set of $\alpha \in S_0$ such that $f \restriction \mathrm{dom}(f) \cap \alpha = a_\alpha$ is stationary.

The existence of such a partition of ω_1 and corresponding $\diamondsuit$ sequence is a consequence of $\diamondsuit$. Indeed, if $\{a_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit$ sequence, then $S_0 = \{\alpha : 0 \in a_\alpha\}$ and $S_1 = \{\alpha : 1 \in a_\alpha\}$ are both stationary, disjoint and $\{a_\alpha \setminus \{i\} : \alpha \in S_i\}$ is a $\diamondsuit_{S_i}$ sequence on $\omega_1 \setminus \{i\}$ for each $i < 2$. Now, by putting together a $\diamondsuit_{S_0}$ sequence and a $\diamondsuit_{S_1}$ sequence one obtains the desired special $\diamondsuit$ sequence. [1]

We want to construct the sets U_α so that a few things happen.

1. For every α, if C_α is closed discrete then $\alpha \notin U_\xi$ for any $\xi \in C_\alpha$. (Since we will make sure that closed discrete sets are countable this assures that X is not a D-space).

2. For each $i < 2$ and each uncountable $T \subseteq S_i$ and each function $f : T \to [\omega_1]^{<\omega}$ such that the range is pairwise disjoint, there is an $\alpha \in S_{1-i}$ such that $f \restriction T \cap \alpha = a_\alpha$ and there is a $D_\alpha \subseteq T \cap \alpha$ that converges to α such that $\{U_{f(\beta)} : \beta \in D_\alpha\}$ covers $S_i \setminus \alpha$.

[1] Thanks to Arnie Miller for pointing this out.

Note that if our space is constructed to be T_2, then (2) implies that D will be closed discrete in S_i, so if we can do (2) then by the previous lemma we will have that both S_0 and S_1 are D-spaces.

So suppose that we are at stage α of the construction and we have constructed $\{U_\beta \cap \alpha : \beta < \alpha\}$. We need to decide whether or not to add α to U_β for each $\beta < \alpha$. Let τ_α be the topology on α generated by the $U_\beta \cap \alpha$'s. Suppose, without loss of generality, that $\alpha \in S_0$. Let $\{\beta_n : n \in \omega\}$ be the set of $\beta \in S_1 \cap \alpha$ for which we have fixed a $D_\beta \subseteq S_0 \cap \beta$ where D_β is closed discrete in $S_0 \cap \alpha$ with respect to the subspace topology determined by τ_α and $\{U_{a_\beta(\xi)} : \xi \in D_\beta\}$ is a cover of $S_0 \cap (\beta, \alpha)$. So we need to assure that α is covered by some set from $\{U_{a_\beta(\xi)} : \xi \in D_\beta\}$.

We also need to consider $a_\alpha : S_1 \cap \alpha \to [\alpha]^{<\omega}$ coding a neighborhood assignment and find $D_\alpha \subseteq S_1 \cap \alpha$ in conjunction with our choice for the neighborhoods for α so that D_α converges to α and so that we will be able to assure that $\{U_{a_\alpha(\xi)} : \xi \in D_\alpha\}$ will cover a tail of the space. Recall that D_α converging to α assures not only that D_α will be closed discrete in S_1 wrt the subspace topology generated by τ_α, but that it will remain closed discrete regardless of how we extend the topology (as long as the final topology is T_2). We begin by proving:

Theorem 0.6. *There exist $\{U^\alpha_\gamma\}_{\gamma \le \alpha}$ and $\varphi_\alpha : (\alpha+1) \to [\mathbb{R}]^{<\omega}$ for $\alpha < \omega_1$ with the following properties:*

***IH**(1)* *$U^\alpha_\gamma \subseteq \alpha + 1$ and $U^\alpha_\alpha = \alpha + 1$ for every $\gamma \le \alpha < \omega_1$, and the range of φ_α is pairwise disjoint for every $\alpha < \omega_1$.*

***IH**(2)* *$U^\alpha_\gamma = U^{\alpha_0}_\gamma \cap (\alpha+1)$ and $\varphi_\alpha = \varphi_{\alpha_0} \restriction (\alpha+1)$ for all $\gamma \le \alpha \le \alpha_0$.*

Let τ_α denote the topology generated by the sets

$$\{U^\alpha_\gamma : \gamma \le \alpha\} \cup \{\varphi_\alpha^{-1}(W) : W \in \mathcal{W}\}$$

as a subbase. Let $U^\alpha_F = \bigcap\{U^\alpha_\gamma : \gamma \in F\}$ for $F \in [\alpha+1]^{<\omega}$.

***IH**(3)* *If C_α is τ_α closed discrete then $\bigcup\{U^\alpha_\gamma : \gamma \in C_\alpha\} \ne \alpha + 1$.*

***IH**(4)* *Let $T_\alpha = \{\beta \le \alpha$: there is a countable elementary submodel $M \prec H(\vartheta)$ for some sufficiently large ϑ such that the requirements (i)-(v) below all hold$\}$.*

- *(i) $M \cap \omega_1 = \beta$,*
- *(ii) $(a_\eta : \eta \in \omega_1), S_0, S_1, (C_\eta : \eta \in \omega_1) \in M$,*
- *(iii) there is a function $\varphi \in M$ such that $\varphi \restriction \beta = \varphi_\beta \restriction \beta$,*
- *(iv) if $\beta \in S_i$ then there is an uncountable $f \in M$ coding a neighborhood assignment to an uncountable subset of S_{1-i} captured by our $\diamondsuit$ sequence at α. I.e., f is such that $dom(f) \subseteq S_{1-i}$ and $f : dom(f) \to [\omega_1]^{<\omega}$ such that*

$f \restriction dom(f) \cap \beta = a_\beta$. Furthermore, $\xi \in U_{f(\xi)}$ for all $\xi \in dom f$.

(v) there is a $\{V_\gamma\}_{\gamma<\omega_1} \in M$ such that $V_\gamma \cap \beta = U^\beta_\gamma \cap \beta$ for all $\gamma < \beta$.

Then for each $i < 2$ and each $\beta \in T_\alpha \cap S_i$ there is a $D_\beta \subseteq dom(a_\beta)$ (independent of α) such that

(a) if $\beta \in T_\alpha$ then both D_β and $\{a_\beta(\xi) : \xi \in D_\beta\}$ converge to β in τ_α (i.e., for each neighborhood V of β, $\{\xi \in D_\beta : \xi \notin V\}$ is finite and $\{\xi \in D_\beta : a_\beta(\xi) \not\subseteq V\}$ is finite), and

(b) if $\beta \in T_\alpha \cap \alpha$ then for every $V \in \tau_\alpha$ with $\beta \in V$ the family

$$\{U^\alpha_{a_\beta(\xi)} : \xi \in D_\beta, a_\beta(\xi) \subseteq V\}$$

is an ω-cover of $(\beta, \alpha] \cap S_{1-i}$.

Let us first show that the theorem implies that the resulting space is hereditarily Lindelöf, not a D-space but each of the subspaces S_0 and S_1 are D-spaces. It clearly follows from IH(1) and IH(2) that the resulting space is a refinement of a T_2 topology, hence is T_2.

To see why each subspace S_i is a D-space, wlog, let us just consider S_0. By Lemma 0.1, it suffices to consider a neighborhood assignment of the form $\{U_{f(\xi)} : \xi \in T\}$ where $T \subseteq S_0$ is uncountable and $f : T \to [\omega_1]^{<\omega}$ is such that the family $\{f(\xi) : \xi \in T\}$ is pairwise disjoint. And it suffices to find a subset of T closed discrete in S_0 whose neighborhoods cover a tail of S_0. So fix such an f and fix a countable elementary submodel containing everything relevant including f and such that $M \cap \omega_1 = \beta$ and $f \restriction (\mathrm{dom}(f) \cap \beta) = a_\beta$. Therefore $\beta \in T_\alpha$ for all $\alpha \geq \beta$. The set D_β given by the Theorem converges to β, and since $D_\beta \subseteq \mathrm{dom}(f) \subseteq S_0$ and since $\beta \in S_1$, it follows, since our topology is T_2, that D_β is closed discrete in S_0. Finally, note that IH(4)(b) implies that $\{U_{f(\xi)} : \xi \in D_\beta\}$ covers $S_0 \setminus \beta$, so by Lemma 0.1, S_0 is a D-space.

Note that this shows that both S_0 and S_1 are hereditarily D-spaces and indeed since the closed discrete sets witnessing D for neighborhood assignments are always countable, it follows that both S_0 and S_1 are hereditarily Lindelöf, so X is hereditarily Lindelöf.

Furthermore, closed discrete subsets of X are countable so IH(3) implies that X is itself not a D-space.

It suffices to prove Theorem 0.6. We construct the sets $\{U_\beta : \beta < \omega_1\}$ by constructing U^α_β for all $\beta < \alpha < \omega_1$ by recursion on α. Suppose we are some stage α and $\{U^\gamma_\beta : \beta < \gamma < \alpha\}$ has been constructed so that for $\gamma < \alpha$ the inductive hypotheses have been preserved. Consider α a limit ordinal. For each $\beta < \alpha$, let

$$\widetilde{U}^\alpha_\beta = \bigcup_{\beta<\gamma<\alpha} U^\gamma_\beta.$$

And let τ_α be the topology generated on α as described in the hypotheses of the theorem.

We let $U^\alpha_\alpha = \alpha + 1$ and we need to decide for each $\beta < \alpha$ whether

- $U^\alpha_\beta = \widetilde{U}^\alpha_\beta$, or
- $U^\alpha_\beta = \widetilde{U}^\alpha_\beta \cup \{\alpha\}$.

Let T_α be as in the inductive hypotheses. Assume that $T_\alpha \cap \alpha \neq \emptyset$ (if it is empty, then the construction is simpler and we leave the reader to check this case). Enumerate as

$$\{(G_n, \beta_n) : n \in \omega\}$$

all pairs (G, β) where $\beta < \alpha$ and $G \in [\alpha \setminus \beta]^{<\omega}$. For each $\beta < \alpha$ let $\{V_n(\beta) : n \in \omega\}$ be a decreasing local neighborhood base at β in the τ_α topology. Since each $\beta < \alpha$ appears infinitely often in the enumeration $\{\beta_n : n \in \omega\}$, the family $\{V_n(\beta_n) : \beta_n = \beta\}$ is a local neighborhood base at β. Also fix an enumeration $\{\alpha_n : n \in \omega\}$ of α and let $\widetilde{\phi}$ denote the function $\bigcup_{\beta<\alpha} \phi_\beta$.

What we do at stage α splits into cases.

Case 1 $\alpha \in T_\alpha$ and C_α is closed discrete in the τ_α topology on α. Fix M witnessing this and fix $f \in M$ such that $f \restriction \mathrm{dom}(f) \cap M = a_\alpha$. Since the domain of f is uncountable, it includes an uncountable subset $E \in M$ such that if we let $g(\eta) = f(\eta) \cup \{\eta\}$ for all $\eta \in \mathrm{dom}(f)$ then $\{g(\eta) : \eta \in E\}$ is pairwise disjoint and

(1) $|g(\eta)| = m$ for all $\eta \in E$ (for some fixed $m \in \omega$), and
(2) for each $\eta \in E$, if $g(\eta) = \{\xi(\eta, i) : i < m\}$ then $|\widetilde{\phi}(\xi(\eta, i))| = k_i$. (for some fixed sequence $(k_i)_{i\in m}$).

Let $N = k_0+...+k_{m-1}$ and let H_ξ denote the N-element set $\bigcup\{\widetilde{\phi}(\xi(\eta, i)) : i < m\}$

We construct now a sequence of finite set $\{F_n : n \in \omega\}$ as follows. Consider (G_0, β_0). Since C_α is closed discrete, and let $W_0 \subseteq V_0(\beta_0)$ be such that $W_0 \cap C_\alpha \subseteq \{\beta_0\}$. Consider now the set $\{a_{\beta_0}(\xi) \subseteq W_0 : \xi \in D_{\beta_0}\}$. By our IH, we know this is an ω-cover of (β_0, α). And M knows this set is countable. Therefore there is a $\xi_0 \in D_{\beta_0}$ such that

(1) $G_0 \subseteq U_{a_{\beta_0}(\xi_0)}$, and
(2) $E' = \{\eta \in E : g(\eta) \subseteq U_{a_{\beta_0}(\xi_0)}\}$ is uncountable.

Now we may fix a $x \in [\mathbb{R}]^N$ which is a complete accumulation point of $\{H_\eta : \eta \in E'\}$ and which is disjoint from $\widetilde{\phi}(\alpha_0)$. Finally fix Q_0 a disjoint union of N rational intervals of measure < 1 containing and separating the points of x and disjoint from $\widetilde{\phi}(\alpha_0)$ with $Q_0^* \in \mathcal{W}$ and let

$$E_0 = \{\eta \in E' : g(\eta) \subseteq \widetilde{\phi}^{-1}(Q_0^*)\}.$$

Since x was a complete accumulation point of $\{H_\eta : \eta \in E'\}$, E_0 is uncountable and since $Q_0 \in M$ it follows that $E_0 \in M$.

Let $F_0 = a_{\beta_0}(\xi_0)$. Note that $G_0 \subseteq U_{F_0}$ and $F_0 \cap C_\alpha = \emptyset$ since $F_0 \subseteq W_0$. And also $\{\eta : a_\alpha(\eta) \cup \{\eta\} \subseteq U_{F_0} \cap \widetilde{\phi}^{-1}(Q_0^*)\} \supseteq E_0 \cap M$ so is infinite.

Proceeding in this fashion it is clear that we can construct sequences $(\xi_i)_{i<\omega}$; $(E_i)_{i<\omega}$; $(F_i)_{i<\omega}$ and $(Q_i)_{i<\omega}$ so that for each $i < \omega$

(1) $\xi_i \in D_{\beta_i}$ and $F_i = a_{\beta_i}(\xi_i)$.
(2) $G_i \subseteq U_{F_i}$ and $F_i \cap C_\alpha = \emptyset$.
(3) $E_i \subseteq E_{i-1}$ is uncountable and $E_i \in M$.
(4) Q_i is a disjoint union of N rational intervals of measure $< 1/i$ and $\overline{Q}_i \subseteq Q_{i-1}$ and $\widetilde{\phi}(\alpha_i) \cap Q_i = \emptyset$.
(5) $E_i \subseteq \{\eta \in \mathrm{dom}(f) : g(\eta) \subseteq \widetilde{\phi}^{-1}(Q_i^*) \cap U_{F_i}\}$

Note that the intersection of the sets Q_i is an N-element subset x_α of $\mathbb{R}$ which is disjoint from $\widetilde{\phi}(\beta)$ for each $\beta < \alpha$. So let ϕ_α extend $\widetilde{\phi}$ by letting $\phi_\alpha(\alpha) = x_\alpha$. Then the range of ϕ_α is pairwise disjoint as required in IH(2).

Now choose $\eta_i \in E_i$ for each i and let $D_\alpha = \{\eta_i : i \in \omega\}$.

For each $\beta \in \bigcup_n F_n$ let $U_\beta^\alpha = \widetilde{U}_\beta^\alpha \cup \{\alpha\}$, and for $\beta \in \alpha \setminus \bigcup_n F_n$ let $U_\beta^\alpha = \widetilde{U}_\beta^\alpha$.

This completes the recursive construction and we need to verify that the inductive hypotheses (1)-(5) are satisfied for α. As noted above, ϕ_α satisfies the requirements of IH(2) and the rest of IH(1) and (2) follow from the construction. IH(3) follows since each $F_n \cap C_\alpha = \emptyset$, so $\alpha \notin U_\xi^\alpha$ for all $\xi \in C_\alpha$. To see that IH(4)(a) holds for α, note first that the following family is a local neighborhood base at α:

$$\{\phi_\alpha^{-1}(Q_n^*) \cap \bigcap_{j<n} U_{F_j}^\alpha : n \in \omega\}.$$

Also note that by construction, for each n and for each $i \geq n$ we have $\eta_i \in E_n$ so that

$$\{\eta_i\} \cup a_\alpha(\eta_i) \subseteq \phi_\alpha^{-1}(Q_i^*) \subseteq \phi_\alpha^{-1}(Q_n^*)$$

and for all $j < i$ we have $E_i \subseteq E_j$ so for all $j \leq n < i$ we have $\{\eta_i\} \cup a_\alpha(\eta_i) \subseteq U_{F_j}^\alpha$. So $\{a_\alpha(\eta_i) : i \in \omega\}$ and D_α both converge to α as required by IH(4)(a).

To verify IH(4)(b), fix $\beta \in T_\alpha \cap \alpha$ and fix a neighborhood V of β in the τ_α topology. Also, fix $G \subseteq (\beta, \alpha]$ finite. Fix now n such that $V_n(\beta) \subseteq V$ and so that $(G_n, \beta_n) = (G \cap \alpha, \beta)$. Then at this stage of the construction we fixed $\xi_n \in D_{\beta_n} = D_\beta$ so that $F_n = a_\beta(\xi_n) \subseteq V_n(\beta) \subseteq V$ and $G \cap \alpha \subseteq U^\alpha_{F_n}$. And $\alpha \in U^\alpha_{F_n}$, so $G \subseteq U^\alpha_{a_\beta(\xi)}$ for some $\xi \in D_\beta$ with $a_\beta(\xi) \subseteq V$ as required.

Case 2 $\alpha \notin T_\alpha$ or C_α is not closed discrete. Then the construction is essentially the same but easier as we do not need to concern ourselves whether the F_n are disjoint from C_α (in the case C_α is not closed discrete) and also we do not need to construct the set D_α in the domain of a_α in the case that $\alpha \notin T_\alpha$. $\square$

Remark: Since each of the subspaces S_0 and S_1 are D-spaces, it follows easily that for any neighborhood assignment of the whole space, there is a discrete subset (the union of a closed discrete subset of S_0 and a closed discrete subset of S_1) whose neighborhoods cover the whole space. This property was called *dually discrete* in [5] where it was asked whether Lindelöf spaces are dually discrete. This question remains open.

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